A curious family of point-covering plane-filling curves

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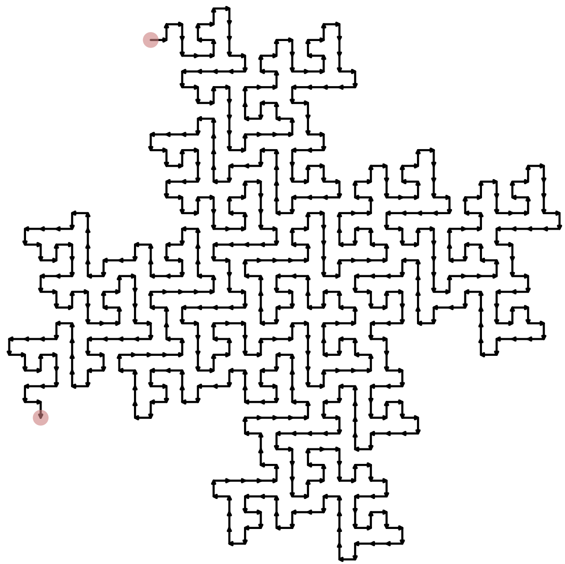
Abstract

We describe a family of plane-filling curves on the square grid with a curious property: The shape of each curve is a lattice tile with 4-fold rotational symmetry. These curves are point-covering, all points of a disk of any size are covered by a high enough iterate of the curve. Our exposition is driven by illustrated examples.

Contents

1 Introduction

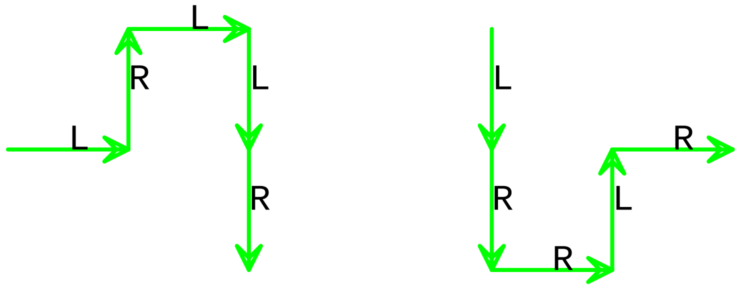
1.1 Point-covering, plane-filling



A point-covering, plane-filling curve on the square grid.

One can cover all points on an arbitrarily large disk with (a high enough iterate of) the curve, it is a *point-covering*, *plane-filling* curve.

1.2 Motifs and order



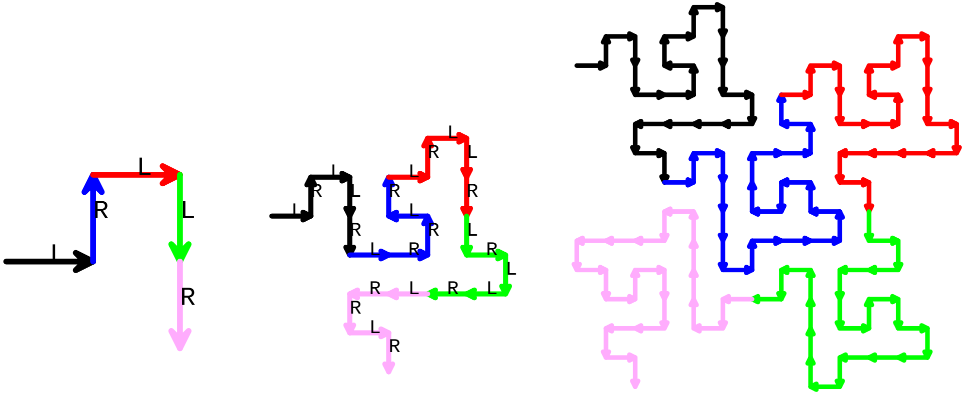
Motifs for the curve: `L` (left) and `R` (right).

The curves can be described by their *motifs*, two curves consisting of 5 edges. We call the numbers of edges in each the *order* of the curve.

To describe curves we use the alphabet { `L`, `R`, `+`, `-`, `0` }. The letters `L` and `R` stand for unit edges and letters `+`, `-`, and `0` for turns by respectively $+90$, -90, and 0 degrees.

The motifs on the left and right are respectively given by `0L+R-L-L0R+` and `-L0R+R+L-R0`. Note that the motifs are mutual reversals: The string `-L0R+R+L-R0` is obtained from `0L+R-L-L0R+` by reversing the string and swapping letters `L` and `R` and letters `+` and `-`.

1.3 Iterates of the curve



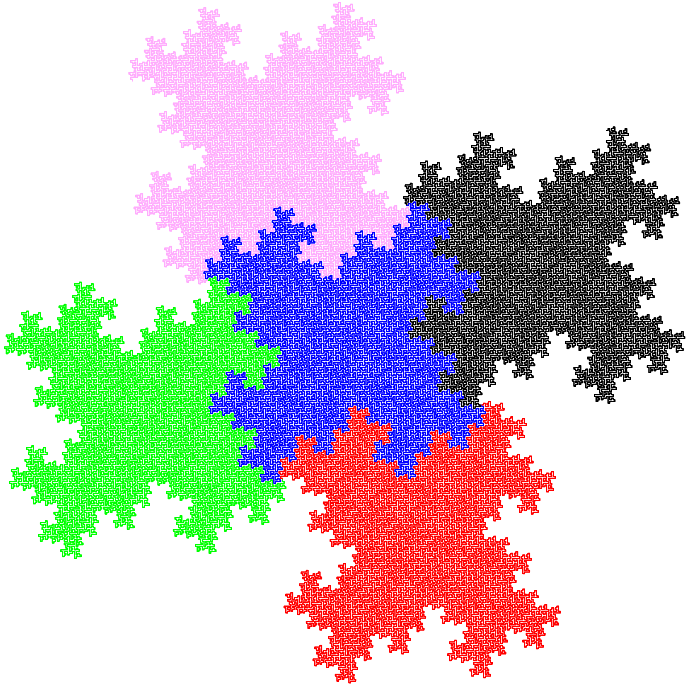
First, second, and third iterate of the curve `L`.

The *iterates* of a curve are obtained by repeatedly replacing edges `L` and `R` by the corresponding motifs.

We color the edges in the first iterate (motif, left) to make the process more clear.

There are 5 copies of the motifs in the second iterate (middle).

The third iterate (right) contains 5 copies of the second, equivalently, 25 copies of the motifs. Our initial image consists of 5 copies of the third iterate, it is the fourth iterate of this curve.



Seventh iterate of the curve. The shape is self-similar, it can be decomposed into five smaller copies of itself.

1.4 Lindenmayer systems

You guessed it, we use Lindenmayer systems to describe our curves.

The L-system for this curve is given by the following maps (from letters to words).

$$\begin{aligned} \texttt{L} &\mapsto \texttt{0L+R-L-L0R+} \\ \texttt{R} &\mapsto \texttt{-L0R+R+L-R0} \\ \texttt{+} &\mapsto \texttt{+} \\ \texttt{-} &\mapsto \texttt{-} \\ \texttt{0} &\mapsto \texttt{0} \end{aligned}$$

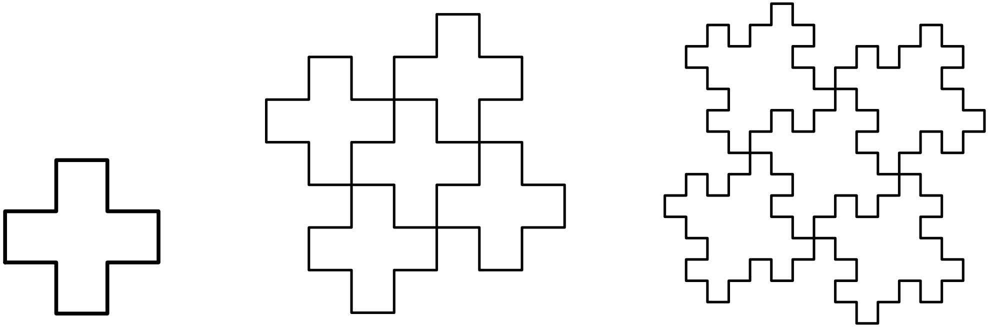
As usual, we only give the non-constant maps, those for `L` and `R`, from here on.

For the images so far we used the axiom `L`.

This curve is due to Douglas McKenna, [5, p. 69-70].

2 The shape of a curve

2.1 A lattice tile consisting of 5 squares

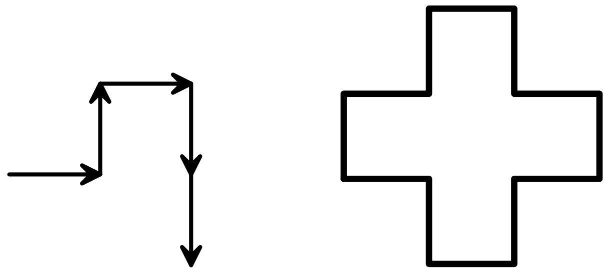


The Greek cross consisting of 5 squares, a lattice tile (left). Combining 5 of those gives the tile shown in the middle. Combining yet again gives the tile on the right.

The only lattice tile consisting of 5 squares with 4-fold rotational symmetry.

We *really* want the shape of our order-5 curve to be this tile!

2.2 Defining the shape of a curve

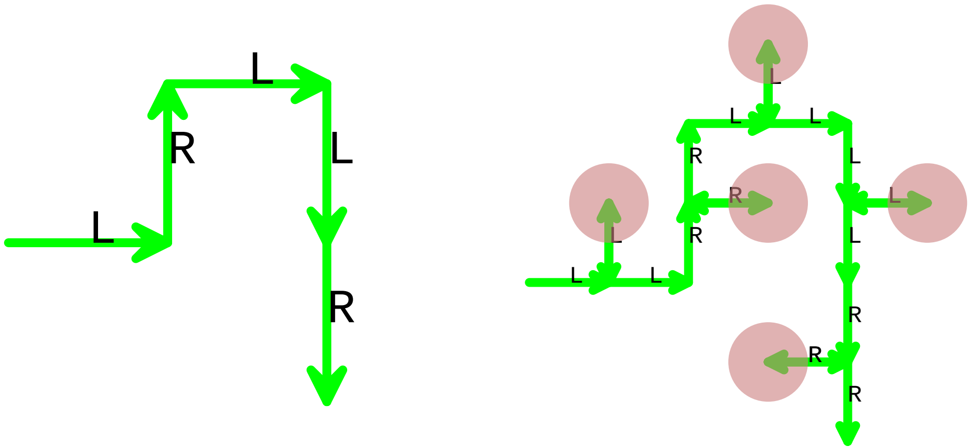


The motif of our curve (left) should have the shape on the right.

Using the points traversed by the curve does not really work.

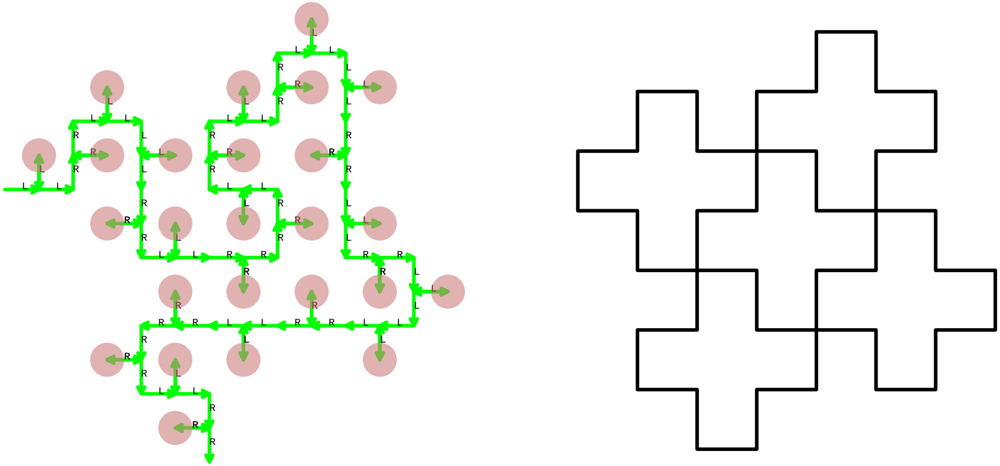
(Dramatic pause)

Excursions (I)



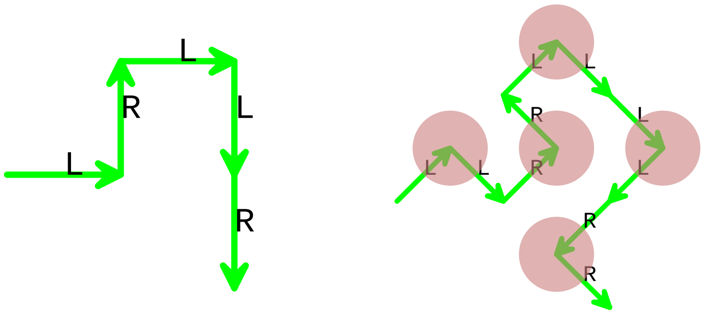
A solution for our problem, excursions: `L` means "left", `R` means "right".

For this image we use the post-processing step
`sed 's/L/L+L-:-L+L/g; s/R/R-R+:+R-R/g;'`
The colon draws a pinkish translucent disk.



Second iterate: looking good.

Excursions (II)



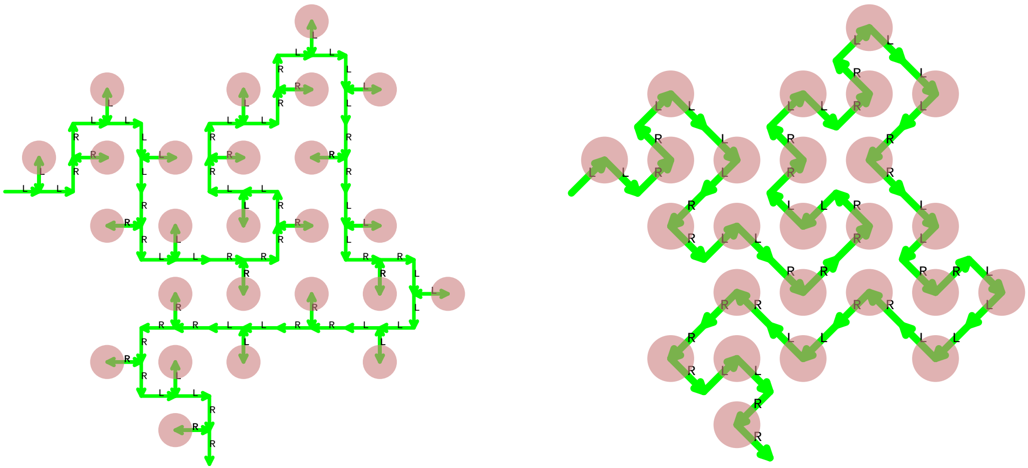
Another way to do the excursions.

The post-processing step here is

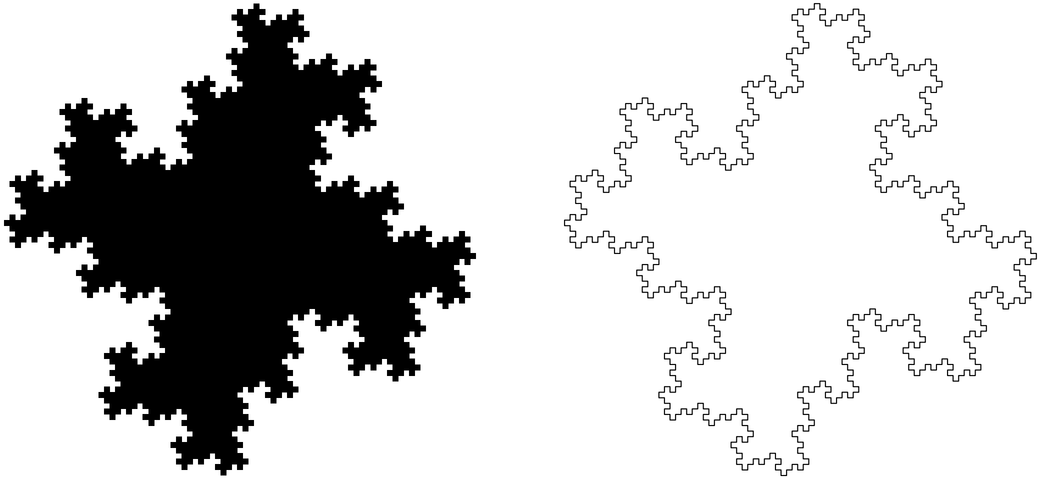
```
sed 's/L/L-:L+/g; s/R/-R+:R/g'
```

This can be used for the search.

Comparing (I) and (II)



Comparing both methods with the second iterate.



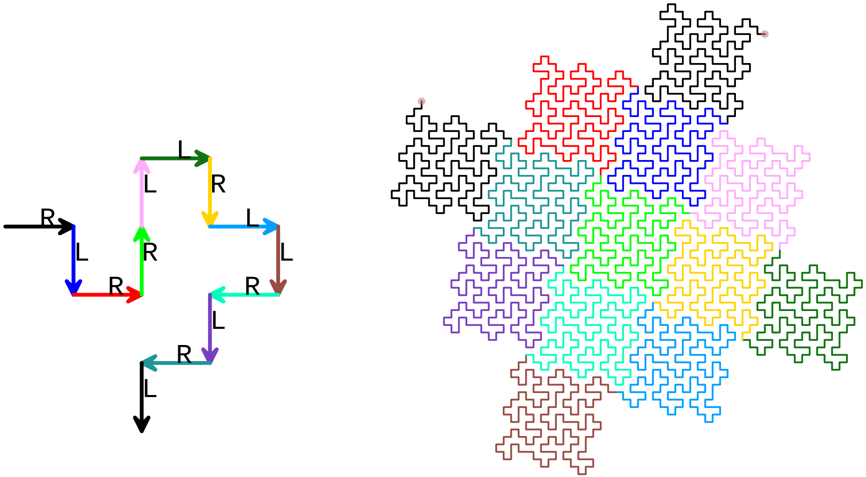
Fifth iterate of the curve (left) and border of the tile (right).

Drawing squares instead of pinkish disks with the fifth iterate gives the image on the left.

Directly generating the border of the tile with the L-system with
map F $\mapsto$ F−F+F   and   axiom F+F+F+F
gives the image on the right.

3 Some examples of such curves

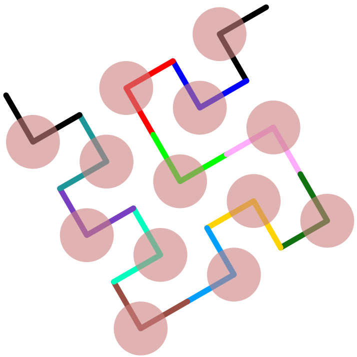
3.1 A curve of order 13

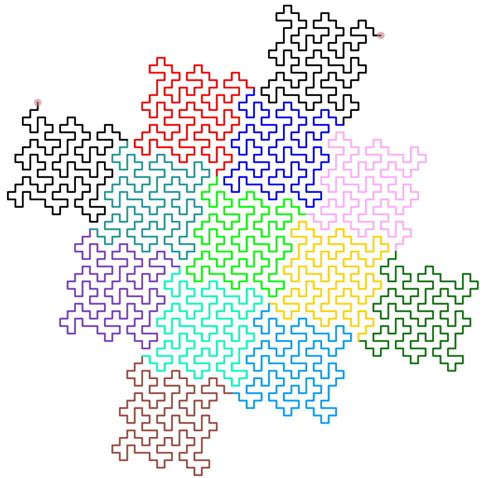


A curve of order 13, first (left) and third (right) iterate. The axiom used is `L`.

The (non-constant) maps of the Lindenmayer system are
`L` $\mapsto$ `0R-L+R+R0L-L-R+L-L-R+L-R+L+` and
`R` $\mapsto$ `-R-L+R-L+R+R-L+R+R0L-L-R+L0`

... and its shape

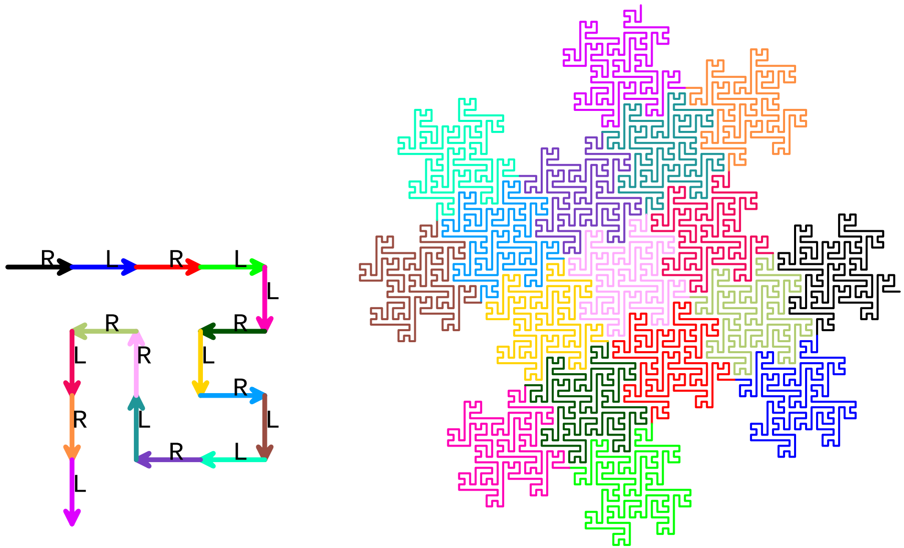




Shape of the curve, indicated by excursions (left) and third iterate (right).

This curve is given by McKenna [p. 70]mckenna-e-curve, see also Ventrella's book [8][p. 168].

3.2 A curve of order 17



A curve of order 17, first (left) and third (right) iterate. The axiom used is `L`.

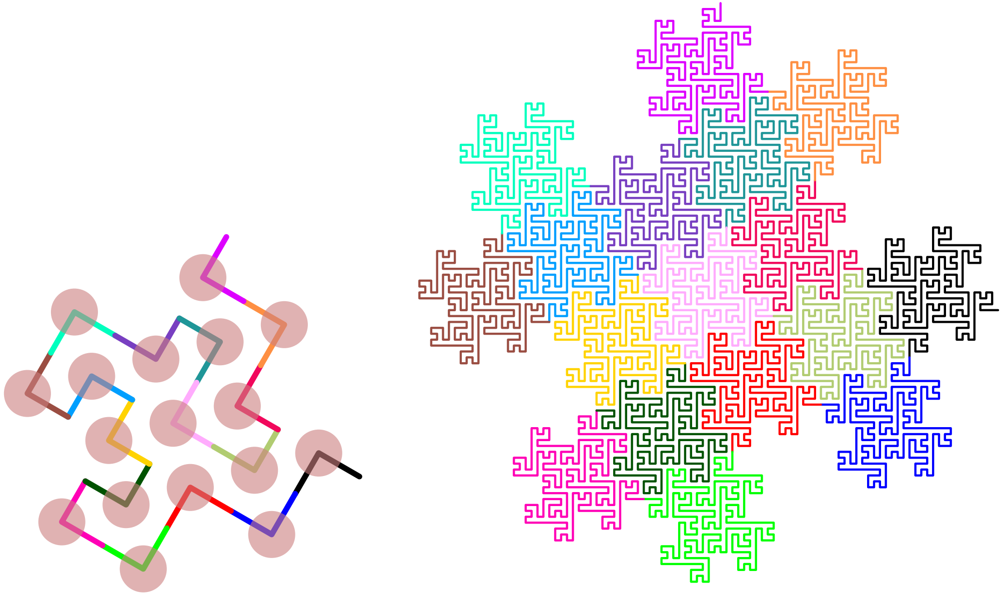
Found in 2019 by Benedikt Repscher [6], pretty sure it is new.

The maps of the Lindenmayer system are

`L` $\mapsto$ `0R0L0R0L-L-R+L+R-L-L0R-L0R+R+L0R0L+` and

`R` $\mapsto$ `-R0L0R-L-L0R+L0R+R+L-R-L+R+R0L0R0L0`

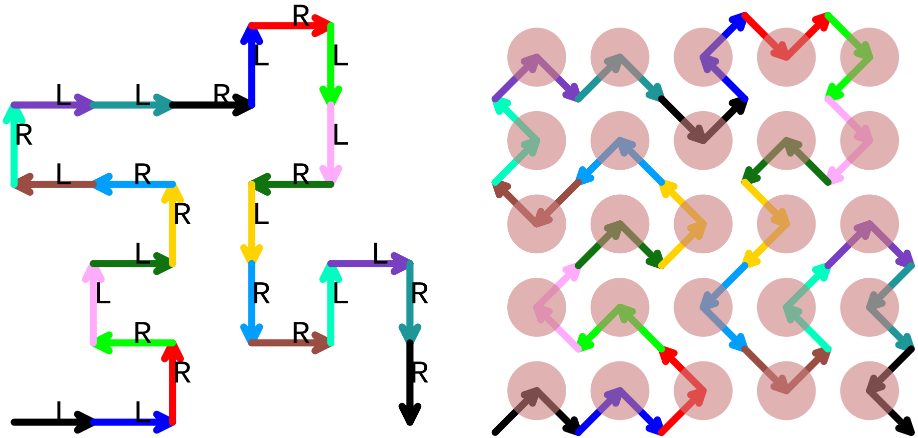
... and its shape



Shape of the curve, indicated by excursions (left) and third iterate (right).

Fourth iterate of the curve.

3.3 A curve of order 25 with square shape



Motif of a curve of order 25 (left) and its shape (right).

Given by Ventrella, [8][p. 194], there attributed to McKenna and Dekking. Appeared 1982 in Dekkings paper "Recurrent Sets" [2][p.93].

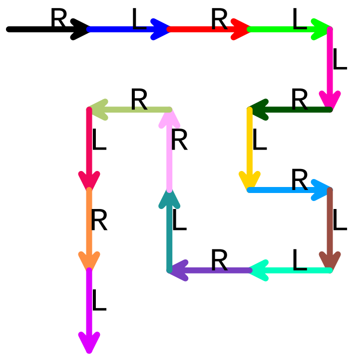
The maps of the Lindenmayer system are

`L` $\mapsto$ `0L0L+R+R-L-L+R+R0L-R-L0L0R+L-R-L0L-R+L0R+R+L-L-R0R+` and

`R` $\mapsto$ `-L0L+R+R-L-L0R-L+R0R+L+R-L0R0R+L+R0L-L-R+R+L-L-R0R0`

4 The search

4.1 Necessary conditions: distance is sum of two squares



First iterate of the curve of order 17 (again, the axiom is `L`).

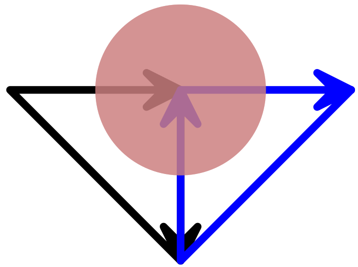
The order R of a curve is the sum of two squares: $R = x^2 + y^2$, $x, y \geq 0$.

R must be odd, so exactly one of x, y must be odd.

x and y are the distances between the (coordinates) of the start- and end-points of the motif.

Also, the net-turn must be zero: The number of (turn-)letters + and − in the map for `L` (and so for `R`) must be equal.

4.2 Necessary conditions: transitions



L+L gives a double point for the shape.

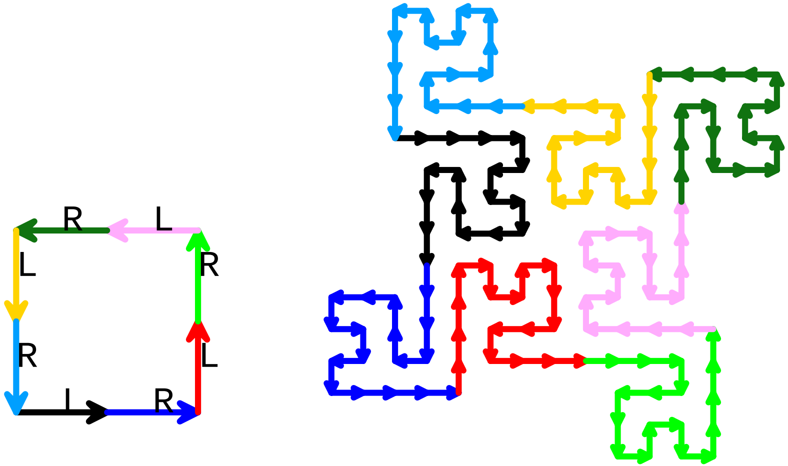
This is not a condidate for the most instructive image in human history.

The transitions L+L and R−R cannot possibly work: both would give a double point for the shape.

All others, L−L, R+R L0L, R0R, L+R, R+L, L−R, and R−L, *must* work.

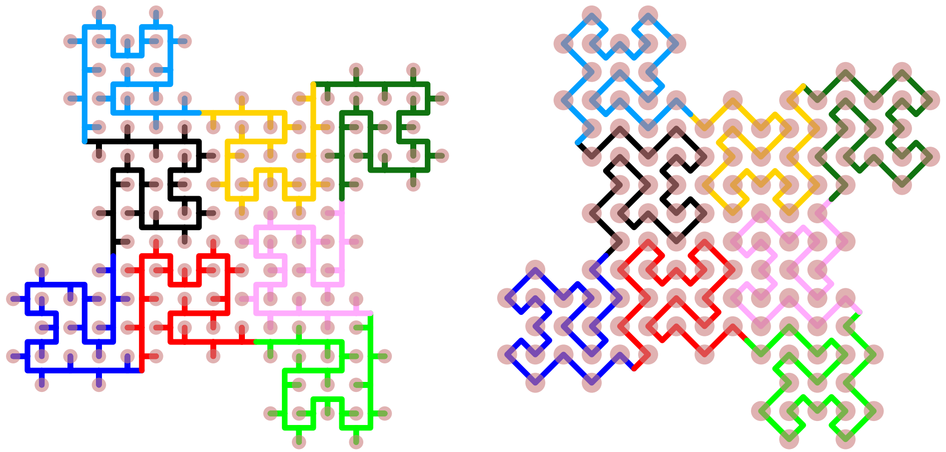
Beware: sometimes these curves are given using L and R with their meaning swapped. Look for transitions L?L and R?R to check.

4.3 Necessary conditions: a tile-condition



Outline of the tile L0R+^{4} (iterate 0, left) and first iterate using our curve of order 17 (right).

The central point of the tile seems to be missing.

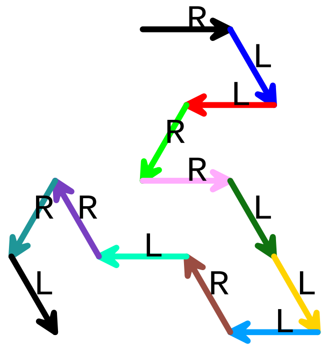


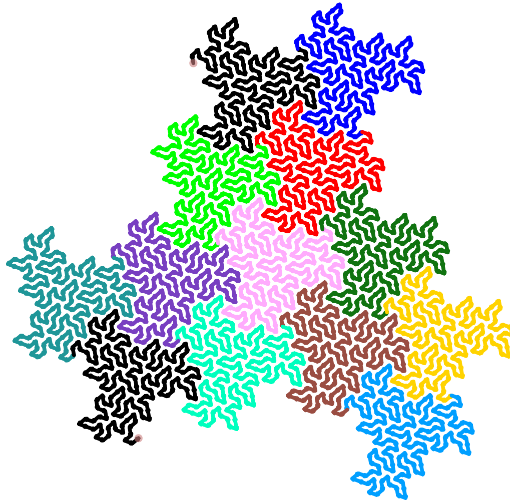
First iterate with our curve of order 17 with shape, both methods shown.

Now nothing is missing in the center.

Michael Hertel's Master's thesis [4] is due June 2026. We (JA, ML, and MH) plan to publish the results on `arxiv.org`.

5 An equivalent family of curves on the triangular grid

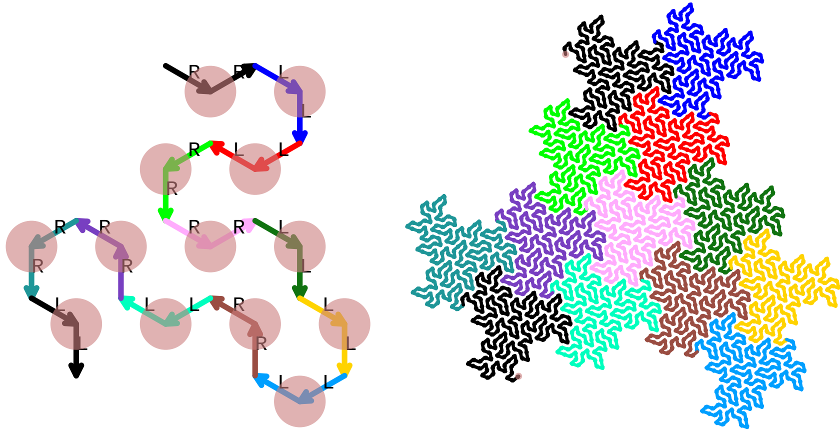




First (left) and second (right) iterate of an order-13 point-covering curve on the triangular grid.

Shown are the iterates for axiom `L`. The maps of the Lindenmayer system are
`L` $\mapsto$ `0R-L--L+R++R-L0L--L-R+L-R++R+L+` and
`R` $\mapsto$ `-R-L--L+R-L+R++R0R+L--L-R++R+L0`

Here the turns (letters + and −) are by 60 degrees.



Shape of the curve via excursions (left) and second (right) iterate.

Arguably on the honeycomb grid.

The post-processing step here is `sed 's/L/L-:L+/g; s/R/-R+:R/g'` (again).

This one is given in Ventrella's book [8][p. 173]. See Akiyama, Fukuda, Ito, Nakamura [1], but also Fukuda, Shimizu, Nakamura [3] (and its two corrigenda).

The smallest such curve is Gosper's, it has order 7.

References

[1] Jin Akiyama, Hiroshi Fukuda, Hiro Ito, Gisaku Nakamura: **Infinite Series of Generalized Gosper Space Filling Curves**, Lecture Notes in Computer Science, vol. 4381, pp. 1-9, (2007). 26

[2] F. M. Dekking: **Recurrent Sets**, Advances in Mathematics, vol. 44, no. 1, pp. 78-104, (1982). URL: `http://dx.doi.org/10.1016/0001-8708(82)90066-4`. 20

[3] Hiroshi Fukuda, Michio Shimizu, Gisaku Nakamura: **New Gosper Space Filling Curves**, in: Proceedings of the International Conference on Computer Graphics and Imaging (CGIM2001), pp. 34-38, (2001). URL: `http://www.researchgate.net/publication/236966236_New_Gosper_Space_Filling_Curves/file/5046351a8209b17cfe.pdf`. 26

[4] Michael Hertel: **A family of point-covering plane-filling curves on the square grid**, Master's thesis, Technische Hochschule Nürnberg, Germany, (2026). 24

[5] Douglas M. McKenna: **SquaRecurves, E-Tours, Eddies, and Frenzies: Basic Families of Peano Curves on the Square Grid**, in Richard K. Guy, Robert E. Woodrow, The Lighter Side of Mathematics: Proceedings of the Eugene Strens Memorial Conference on Recreational Mathematics and its History, Mathematical Association of America, pp. 49-73, (1994). 7

[6] Benedikt Repscher: **Search for Point-covering plane-filling curves created by Lindenmayer systems**, Master's thesis, Technische Hochschule Nürnberg, Germany, (2019). 17

[7] Martin Strauß: **Searching for point-covering curves on the square grid**, Master's thesis, Technische Hochschule Nürnberg, Germany, (2019).

[8] Jeffrey J. Ventrella: **Brain-Filling Curves: A Fractal Bestiary**, LuLu.com, (2012). URL: https://archive.org/download/BrainfillingCurves-AFractalBestiary/. 16, 20, 26