ArcTan Related Formulas

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Warning: the stuff here is a bit cryptic.

general

$$\arctan\left(\frac{1}{x}\right) = \arctan\left(\frac{1}{x+d}\right) + \arctan\left(\frac{1}{x+\frac{x^2+1}{d}}\right)$$

$$\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{y}\right) = \arctan\left(\frac{1}{\frac{x\,y-1}{x+y}}\right)$$

$$\arctan\left(\frac{1}{z}\right) = \arctan\left(\frac{1}{x_1}\right) + \arctan\left(\frac{1}{x_2}\right) + \ldots + \arctan\left(\frac{1}{x_n}\right) \iff z = \frac{Re(\prod_{k=1}^{n}(x_k+i))}{Im(\prod_{k=1}^{n}(x_k+i))}$$

$$\iff (z^2+1) = \frac{\prod_{k=1}^{n}(x_k^2+1)}{\left(Im(\prod_{k=1}^{n}(x_k+i))\right)^2}$$

$$\arctan\left(\frac{1}{x}\right) = \arctan\left(\frac{1}{y}\right) + \arctan\left(\frac{1}{z}\right) \iff (x^2+1) = \frac{(y^2+1)(z^2+1)}{(y+z)^2} = (y-x)(z-x)$$

$$\frac{k\,\pi}{4} = \arctan\frac{b_1}{a_1} + \arctan\frac{b_2}{a_2} + \ldots + \arctan\frac{b_n}{a_n} \iff (a_1+i\,b_1)(a_2+i\,b_2)\ldots(a_n+i\,b_n)(1-i)^k \quad \in \mathbb{R}$$

$$\frac{k\,\pi}{4} = \arctan\left(\frac{1}{a_1}\right) + \arctan\left(\frac{1}{a_2}\right) + \ldots + \arctan\left(\frac{1}{a_n}\right) \iff (a_1+i)(a_2+i)\ldots(a_n+i)(1-i)^k \quad \in \mathbb{R}$$

$$\frac{k\,\pi}{4} = m_1\arctan\left(\frac{1}{a_1}\right) + m_2\arctan\left(\frac{1}{a_2}\right) + \ldots + m_n\arctan\left(\frac{1}{a_n}\right)$$

$$\iff (a_1+i)^{m_1}(a_2+i)^{m_2}\ldots(a_n+i)^{m_n}(1-i)^k \quad \in \mathbb{R}$$

$$a>b, \quad gcd(a,b)=1, \quad a\,\alpha+b\,\beta=1 \iff \arctan\left(\frac{a}{b}\right) = \arctan\left(\frac{-\beta}{\alpha}\right) + \arctan\left(\frac{1}{a\,\beta-b\,\alpha}\right)$$

proof:

$$r := a\,\beta - b\,\alpha, \quad \arctan\left(\frac{1}{z}\right) = \arctan\left(\frac{1}{a/b}\right) + \arctan\left(\frac{1}{r}\right)$$

$$\iff z = \frac{(a/b)\,r-1}{(a/b)+r} = \frac{a(\alpha\,b-\beta\,a)-b}{a+b(\alpha\,b-\beta\,a)} = \frac{a(\alpha\,b-\beta\,a)-b(1)}{a(1)+b(\alpha\,b-\beta\,a)}$$

$$= \frac{a(\alpha\,b-\beta\,a)-b(\alpha\,a+\beta\,b)}{a(\alpha\,a+\beta\,b)+b(\alpha\,b-\beta\,a)} = \frac{-a^2\,\beta-b^2\,\beta}{\alpha\,a^2+\alpha\,b^2} = \frac{-\beta}{\alpha}$$

definitions

$$q(n) := \frac{3n+n^3}{2} \qquad F_n := Fibonacci(n) = 1,1,2,3,5,8,13,21,34,55,...$$

$$jt(a_0,a_1,m_1,m_2,0) := a0, \qquad jt(a_0,a_1,m_1,m_2,1) := a1,$$

$$jt(a_0,a_1,m_1,m_2,n) := m_1\, jt(a_0,a_1,m_1,m_2,n-1) + m_2\, jt(a_0,a_1,m_1,m_2,n-2)$$

$$jt(a_0,a_1,m_1,m_2,n) = C_1\, x_1^n + C_2\, x_2^n \quad \text{where:} \quad x_{1,2}^2 = m_1\, x_{1,2} + m_2, \quad C_1 + C_2 = a_0, \quad C_1\, x_1 + C_2\, x_2 = a_1$$

semi-crypt

$$\arctan\left(\frac{1}{x}\right) = \arctan\left(\frac{1}{x+1}\right) + \arctan\left(\frac{1}{x^2+x+1}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{1}\right) = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) =$$

$$= \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) + \arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{31}\right) + \arctan\left(\frac{1}{43}\right) + \arctan\left(\frac{1}{57}\right) + ...$$

$$\arctan\left(\frac{1}{2x+1}\right) = \arctan\left(\frac{1}{2x+3}\right) + \arctan\left(\frac{1}{2x^2+4x+2}\right)$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) =$$

$$= \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{8}\right) + \arctan\left(\frac{1}{18}\right) + \arctan\left(\frac{1}{50}\right) + \arctan\left(\frac{1}{98}\right) + \arctan\left(\frac{1}{128}\right) + \dots$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \sum_{n=0}^{\infty} \left(\arctan\left(\frac{1}{5+11x+5x^2}\right) + \arctan\left(\frac{1}{5+9x+5x^2}\right)\right)$$

$$= \sum_{n=0}^{\infty} \left(\arctan\left(\frac{1}{\frac{(5x+2)^2+1}{5} + (5x+2)}\right) + \arctan\left(\frac{1}{\frac{(5x+3)^2+1}{5} + (5x+3)}\right)\right)$$

<u>continued fraction related ...</u>

partial quotients $P_n/Q_n =$ of $\sqrt{1+x^2}$:

$$P_n = \frac{(x+\sqrt{1+x^2})^n + (x-\sqrt{1+x^2})^n}{2}$$

$$Q_n = \frac{(x+\sqrt{1+x^2})^n - (x-\sqrt{1+x^2})^n}{2\sqrt{1+x^2}}$$

$$\arctan\left(\frac{1}{\sinh((2k-1)\phi)}\right) = 2\arctan\left(\frac{1}{\sinh(2k\phi)/(\cosh(\phi))}\right) - \arctan\left(\frac{1}{\sinh((2k+1)\phi)}\right)$$

integer denominators if $\phi = \text{arcsinh(x)}$ integer x

$$\arctan\left(\frac{1}{x}\right) = 2\sum_{n=1}^{\infty}(-1)^n \arctan\left(\frac{1}{jt(0,1,2x,1,2n)}\right) =$$

$$= 2\left(\arctan\left(\frac{1}{2x}\right) - \arctan\left(\frac{1}{4x+8x^3}\right) + \arctan\left(\frac{1}{6x+32x^3+32x^5}\right) - + \ldots\right) =$$

$$= 2\sum_{n=1}^{\infty}(-1)^n \arctan\left(\frac{1}{\sinh(2n\, arcosh(x))/\sqrt{x^2+1}}\right)$$

$$\frac{\pi}{8} = \sum_{n=1}^{\infty}(-1)^n \arctan\left(\frac{1}{jt(0,1,2,1,2n)}\right) =$$

$$= \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{1}{12}\right) + \arctan\left(\frac{1}{70}\right) - \arctan\left(\frac{1}{408}\right) + \arctan\left(\frac{1}{2378}\right) - + \ldots$$

$$\frac{\pi}{8} = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{jt(1,1,2,1,2n)}\right) =$$

$$= \sum_{n=1}^{\infty}(-1)^n \arctan\left(\frac{1}{(1-\sqrt{2})^{2n} + (1+\sqrt{2})^{2n}}\right) =$$

$$= \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{17}\right) + \arctan\left(\frac{1}{99}\right) + \arctan\left(\frac{1}{577}\right) + \arctan\left(\frac{1}{3363}\right) + \ldots$$

$$f(n) := jt(0,1,2,1,2n)\, jt(0,1,2,1,2n+1)^2 + jt(1,1,2,1,2n+1)$$

$$eg. \quad f(0) = 2\cdot 5^2 + 7 = 57, \quad f(1) = 12\cdot 29^2 + 41 = 10133, \quad \ldots$$

$$\frac{\pi}{8} = \sum_{n=1}^{\infty} 3^n \arctan\left(\frac{1}{jt(0,1,2,1,2n+1)}\right) - \sum_{n=1}^{\infty} 3^n \arctan\left(\frac{1}{f(n)}\right)$$

$$= \arctan\left(\frac{1}{5}\right) + 3\left(\arctan\left(\frac{1}{29}\right) + 3\left(\arctan\left(\frac{1}{169}\right) + 3\left(\arctan\left(\frac{1}{5741}\right) + \ldots\right)\right)\right) -$$

$$- \left(\arctan\left(\frac{1}{57}\right) + 3\left(\arctan\left(\frac{1}{10133}\right) + 3\left(\arctan\left(\frac{1}{1999509}\right) + \ldots\right)\right)\right)$$

$$\arctan\left(\frac{1}{P_n}\right) = 4\arctan\left(\frac{1}{Q_{n+2}}\right) + \arctan\left(\frac{1}{P_{n+4}}\right) - 2\arctan\left(\frac{1}{q(Q_{n+2})}\right) \text{(n odd)}$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{P_1}\right) =$$

$$= 4\,\left(\arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{169}\right) + \arctan\left(\frac{1}{5741}\right) + \ldots\right) -$$

$$-2\,\left(\arctan\left(\frac{1}{q(5)}\right) + \arctan\left(\frac{1}{q(169)}\right) + \arctan\left(\frac{1}{q(5741)}\right) + \ldots\right)$$

$$\text{with } f(n) := jt(0,1,2,1,n+3)\,jt(0,1,2,1,n+2)^2 + jt(1,1,2,1,n+2)$$

$$\arctan\left(\frac{1}{jt(1,1,2,1,n)}\right) = 5\arctan\left(\frac{1}{jt(1,1,2,1,n+2)}\right) + 2\arctan\left(\frac{1}{jt(1,1,2,1,n+3)}\right) -$$

$$-\arctan\left(\frac{1}{q(jt(0,1,2,1,n+2))}\right) + 2\arctan\left(\frac{1}{f(n)}\right) \quad \text{(n odd)}$$

$$\frac{\pi}{4} = 2\arctan\left(\tfrac{1}{3}\right) + \arctan\left(\tfrac{1}{7}\right) = 2\arctan\left(\tfrac{1}{3}\right) + 2\arctan\left(\tfrac{1}{17}\right) + \arctan\left(\tfrac{1}{41}\right) =$$

$$= \left(12\arctan\left(\tfrac{1}{17}\right) + 5\arctan\left(\tfrac{1}{41}\right)\right) - 2\arctan\left(\tfrac{1}{70}\right) - 4\arctan\left(\tfrac{1}{307}\right) =$$

$$= \left(70\arctan\left(\tfrac{1}{99}\right) + 29\arctan\left(\tfrac{1}{239}\right)\right) - 2\arctan\left(\tfrac{1}{70}\right) - 4\arctan\left(\tfrac{1}{307}\right) - 12\arctan\left(\tfrac{1}{12238}\right) - 24\arctan\left(\tfrac{1}{58911}\right) =$$

$$= \left(408\arctan\left(\tfrac{1}{577}\right) + 169\arctan\left(\tfrac{1}{1393}\right)\right) - (2\arctan\left(\tfrac{1}{70}\right) + 4\arctan\left(\tfrac{1}{307}\right) + 12\arctan\left(\tfrac{1}{12238}\right) + 24\arctan\left(\tfrac{1}{58911}\right) + 70\arctan\left(\tfrac{1}{\ldots}\right) + 140\arctan\left(\tfrac{1}{\ldots}\right)$$

$$\frac{\pi}{4} = 2(\sqrt{2}-1) - \sum_{n=1}^{\infty}\left(jt(0,1,2,1,2n)\arctan\left(\frac{1}{q(jt(0,1,2,1,2n+1))}\right) + 2\,jt(0,1,2,1,2n)\arctan\left(\frac{1}{f(2n+1)}\right)\right)$$

$$\frac{\pi}{4} = 3\,\sum_{n=1}^{\infty}\arctan\left(\frac{1}{jt(1,1,3,1,2n)}\right) - \sum_{n=1}^{\infty}\arctan\left(\frac{1}{jt(1,1,3,1,2n)}\right) =$$

$$= 3\left(\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{43}\right) + \arctan\left(\frac{1}{469}\right) + \arctan\left(\frac{1}{5116}\right) + \ldots\right) -$$

$$-\left(\arctan\left(\frac{1}{q(4)}\right) + \arctan\left(\frac{1}{q(43)}\right) + \arctan\left(\frac{1}{q(469)}\right) + \arctan\left(\frac{1}{q(5116)}\right) + \ldots\right) =$$

$$= 3\left(\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{1}{43}\right) + \arctan\left(\frac{1}{469}\right) + \arctan\left(\frac{1}{5116}\right) + \ldots\right) -$$

$$-\left(\arctan\left(\frac{1}{38}\right) + \arctan\left(\frac{1}{39818}\right) + \arctan\left(\frac{1}{51581558}\right) + \ldots\right)$$

$$\arctan\left(\frac{1}{x}\right) = 2\arctan\left(\frac{1}{2x}\right) - \arctan\left(\frac{1}{3x+4x^3}\right)$$

$$\frac{\pi}{4} = 1 - \sum_{n=0}^{\infty} 2^n\arctan\left(\frac{1}{3\,2^n + 4\,2^{3n}}\right) = 1 - \sum_{n=0}^{\infty} 2^n\arctan\left(\frac{1}{q(2^{n+1})}\right)$$

$$= 1 - \left(\arctan\left(\frac{1}{7}\right) + 2\arctan\left(\frac{1}{38}\right) + 4\arctan\left(\frac{1}{268}\right) + 8\arctan\left(\frac{1}{2072}\right) + 16\arctan\left(\frac{1}{16432}\right) + \ldots\right)$$

<u>fibonaccoidal</u>

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_{2n+1}}\right) =$$

$$= \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{34}\right) + \arctan\left(\frac{1}{89}\right) + \arctan\left(\frac{1}{233}\right) + \dots$$

$$\boxed{\arctan\left(\frac{1}{F_{2n}}\right) = 3\arctan\left(\frac{1}{F_{2n+2}}\right) - \arctan\left(\frac{1}{F_{2n+4}}\right) - \arctan\left(\frac{1}{q(F_{2n+2})}\right)}$$

$\Rightarrow$

$$\boxed{\frac{\pi}{4} = \frac{3\sqrt{5}-5}{2} - \sum_{n=1}^{\infty} F_{2n} \arctan\left(\frac{1}{q(F_{2n+2})}\right)}$$

$$= \frac{\pi}{4} = \frac{3\sqrt{5}-5}{2} - \left(1\arctan\left(\frac{1}{18}\right) + 3\arctan\left(\frac{1}{268}\right) + 8\arctan\left(\frac{1}{4662}\right) + 21\arctan\left(\frac{1}{83270}\right) + \dots\right)$$

$$\arctan\left(\frac{1}{F_{2n}}\right) = \sum_{k=0}^{N} \arctan\left(\frac{1}{F_{2n+2k+1}}\right) + \arctan\left(\frac{1}{2n+2N+1}\right)$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} n\left[\arctan\left(\frac{1}{F_{2n+2}}\right) - \arctan\left(\frac{1}{q(F_{2n+2})}\right)\right] =$$

$$= \sum_{n=1}^{\infty} n\left[\sum_{m=1}^{\infty} m\left[\arctan\left(\frac{1}{F_{2n+2m+2}}\right) - \arctan\left(\frac{1}{q(F_{2n+2m+2})}\right)\right] - \arctan\left(\frac{1}{q(F_{2n+2})}\right)\right] = \text{etc.}$$

<u>cryptoidal</u>

$$\boxed{\arctan\frac{2}{x+\sqrt{x^2-4}}=\sum_{n=1}^{\infty}\arctan\frac{1}{x\,jt(0,1,x,-1,n)^2}}$$

for x=2:

$$\arctan\frac{2}{2+\sqrt{2^2-4}}=\arctan 1=\frac{\pi}{4}=\sum_{n=1}^{\infty}\arctan\left(\frac{1}{2\,n^2}\right)$$

for x=4:

$$\arctan\frac{2}{4+\sqrt{4^2-4}}=\arctan\left(\frac{1}{2+\sqrt{3}}\right)=\frac{\pi}{12}$$

$$=\sum_{n=1}^{\infty}\arctan\frac{1}{4\,jt(0,1,4,-1,n)^2}$$

$$=\arctan\left(\frac{1}{4}\right)+\arctan\left(\frac{1}{64}\right)+\arctan\left(\frac{1}{900}\right)+\arctan\left(\frac{1}{12544}\right)+\arctan\left(\frac{1}{174724}\right)+\arctan\left(\frac{1}{2433600}\right)+...$$

$$\arctan\left(\frac{1}{2x^2}\right)=2\arctan\left(\frac{1}{4x^2-2x+1}\right)-\arctan\left(\frac{1}{4x^2+2x+1}\right)$$

$$\Rightarrow\frac{\pi}{4}=\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{4n^2-2n+1}\right)+\arctan\left(\frac{1}{4n^2+2n+1}\right)\right]$$

$$\arctan\left(\frac{1}{2x^2}\right)=2\arctan\left(\frac{1}{4x^2+2x+1}\right)-\arctan\left(\frac{1}{2x^3+x}\right)+\arctan\left(\frac{1}{4x^3+3x}\right)$$

$$\arctan\left(\frac{1}{2x^2}\right)=2\arctan\left(\frac{1}{4x^2-2x+1}\right)+\arctan\left(\frac{1}{2x^3+x}\right)-\arctan\left(\frac{1}{4x^3+3x}\right)$$

$$\Rightarrow\frac{\pi}{4}=\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{4n^2+2n+1}\right)+\arctan\left(\frac{1}{2n^3+n}\right)-\arctan\left(\frac{1}{4n^3+3n}\right)\right]$$

$$\boxed{\frac{\pi}{6}=\sum_{n=0}^{\infty}\arctan\left(\frac{1}{2\,jt(1,3,4,-1,n)^2}\right)}$$

$$=\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{18}\right)+\arctan\left(\frac{1}{242}\right)+\arctan\left(\frac{1}{3362}\right)+\arctan\left(\frac{1}{46818}\right)+\arctan\left(\frac{1}{652082}\right)+...$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\arctan\left(\frac{1}{2n^2}\right)=\text{(by changing the order of summation)}$$

$$\boxed{\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{2n+1}}\zeta(4n+2)}$$