Remarks on the computation of π

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[1]This document is enclosed in the hfloat-package which is online at http://www.jjj.de/hfloat/. Note that the part about arithmetical algorithms was merged into the fxtbook, online at http://www.jjj.de/fxt/.

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Chapter 1

Remarks on the computation of π

1.1 Iterations for π

For general forms of the examples given here see J. & P. Borwein's book [10] and their papers.

For some iterations an operation count (in units of full precision multiplications) is given. Operations different from multiplication are counted as follows:
1 squaring = 2/3 mult.
1 division = 4 mult.
1 inverse sqrt = 4 mult.
1 sqrt = 5 mult.

full prec mult → efficiency measure

2.order iteration, cf. `src/pi/pi2nd.cc`:

$$\begin{aligned}
y_0 &= \frac{1}{\sqrt{2}} && (1.1)\\
a_0 &= \frac{1}{2} && (1.2)\\
y_{k+1} &= \frac{1-(1-y_k^2)^{1/2}}{1+(1-y_k^2)^{1/2}} \quad \to 0+ && (1.3)\\
&= \frac{(1-y_k^2)^{-1/2}-1}{(1-y_k^2)^{-1/2}+1} && (1.4)\\
a_{k+1} &= a_k\,(1+y_{k+1})^2 - 2^{k+1}\,y_{k+1} \quad \to \frac{1}{\pi} && (1.5)\\
a_k - \pi^{-1} &\le 16\cdot 2^{k+1}\,e^{-2^{k+1}\,\pi} && (1.6)
\end{aligned}$$

1.4 shows how to save 1 multiplication per step.

Operations per step: 1 inverse sqrt, 1 division, 2 squarings, 1 multiplication.

Quartic (4.order) iterations, cf. `src/pi/pi4th.cc`:
variant $r = 4$:

$$\begin{aligned}
y_0 &= \sqrt{2}-1 && (1.7)\\
a_0 &= 6-4\sqrt{2} && (1.8)\\
y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \to 0+ && (1.9)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} && (1.10)\\
a_{k+1} &= a_k\,(1+y_{k+1})^4 - 2^{2k+3}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} && (1.11)\\
&= a_k\,((1+y_{k+1})^2)^2 - 2^{2k+3}\,y_{k+1}\,((1+y_{k+1})^2-y_{k+1}) && (1.12)\\
0 &< a_k - \pi^{-1} \le 16\cdot 4^n\,2\,e^{-4^n\,2\,\pi} && (1.13)
\end{aligned}
$$

Identities 1.10 and 1.12 show how to save operations.

Operations per step: 1 inverse 4th root, 1 division, 2 squarings, 1 multiplication.

variant $r = 16$:

$$
\begin{aligned}
y_0 &= \frac{1-2^{-1/4}}{1+2^{-1/4}} && (1.14)\\
a_0 &= \frac{8/\sqrt{2}-2}{(2^{-1/4}+1)^4} && (1.15)\\
y_{k+1} &= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ && (1.16)\\
a_{k+1} &= a_k\,(1+y_{k+1})^4 - 2^{2k+4}\,y_{k+1}\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} && (1.17)\\
0 &< a_k - \pi^{-1} \le 16\cdot 4^n\,4\,e^{-4^n\,4\,\pi} && (1.18)
\end{aligned}
$$

Same operation count as before, but this variant gives approximately twice as much precision after the same number of steps.

The general form of the quartic iterations (1.7 and 1.14) is

$$
\begin{aligned}
y_0 &= \sqrt{\lambda^*(r)} && (1.19)\\
a_0 &= \alpha(r) && (1.20)\\
y_{k+1} &= \frac{(1-y_k^4)^{-1/4}-1}{(1-y_k^4)^{-1/4}+1} \quad \to 0+ && (1.21)\\
a_{k+1} &= a_k\,(1+y_{k+1})^4 - 2^{2k+2}\,\sqrt{r}\,y_k\,(1+y_{k+1}+y_{k+1}^2) \quad \to \frac{1}{\pi} && (1.22)\\
0 &< a_k - \pi^{-1} \le \quad 16\cdot 4^n\,\sqrt{r}\,e^{-4^n\,\sqrt{r}\,\pi} && (1.23)
\end{aligned}
$$

Cf. [10], p.170f.

AGM variant 1, cf. `src/pi/piagm.cc`:

$$
\begin{aligned}
a_0 &= 1 && (1.24)\\
b_0 &= \frac{1}{\sqrt{2}} && (1.25)\\
p_n &= \frac{2\,a_{n+1}^2}{1-\sum_{k=0}^{n} 2^k\,c_k^2} \quad \to \pi && (1.26)\\
\pi - p_n &= \frac{\pi^2\,2^{n+4}\,e^{-\pi\,2^{n+1}}}{AGM^2(a_0,b_0)} && (1.27)
\end{aligned}
$$

cf. also `src/pi/piagm.cc`.

AGM variant 3fast, cf. `src/pi/piagm3.cc`:

$$
a_0 = 1 \qquad (1.28)
$$

$$b_0 = \frac{\sqrt{6}+\sqrt{2}}{4} \tag{1.29}$$

$$p_n = \frac{2\,a_{n+1}^2}{\sqrt{3}\,(1-\sum_{k=0}^{n} 2^k\,c_k^2)-1} \quad \to \pi \tag{1.30}$$

$$\pi - p_n < \frac{\sqrt{3}\,\pi^2\,2^{n+4}\,e^{-\sqrt{3}\,\pi\,2^{n+1}}}{AGM^2(a_0,b_0)} \tag{1.31}$$

AGM variant 3slow, cf. `src/pi/piagm3.cc`:

$$a_0 = 1 \tag{1.32}$$

$$b_0 = \frac{\sqrt{6}-\sqrt{2}}{4} \tag{1.33}$$

$$p_n = \frac{6\,a_{n+1}^2}{\sqrt{3}\,(1-\sum_{k=0}^{n} 2^k\,c_k^2)+1} \quad \to \pi \tag{1.34}$$

$$\pi - p_n < \frac{\frac{1}{\sqrt{3}}\,\pi^2\,2^{n+4}\,e^{-\frac{1}{\sqrt{3}}\,\pi\,2^{n+1}}}{AGM(a_0,b_0)^2} \tag{1.35}$$

Derived AGM iteration (2.order), cf. `src/pi/pideriv.cc`:

$$x_0 = \sqrt{2} \tag{1.36}$$

$$p_0 = 2+\sqrt{2} \tag{1.37}$$

$$y_1 = 2^{1/4} \tag{1.38}$$

$$x_{k+1} = \frac{1}{2}\left(\sqrt{x_k}+\frac{1}{\sqrt{x_k}}\right) \quad (k \geq 0) \quad \to 1+ \tag{1.39}$$

$$y_{k+1} = \frac{y_k\,\sqrt{x_k}+\frac{1}{\sqrt{x_k}}}{y_k+1} \quad (k \geq 1) \quad \to 1+ \tag{1.40}$$

$$p_{k+1} = p_k\,\frac{x_k+1}{y_k+1} \quad (k \geq 1) \quad \to \pi + \tag{1.41}$$

$$p_k - \pi = 10^{-2^{k+1}} \tag{1.42}$$

Cubic AGM, from [17], cf. `src/pi/picubagm.cc`:

$$a_0 = 1 \tag{1.43}$$

$$b_0 = \frac{\sqrt{3}-1}{2} \tag{1.44}$$

$$a_{n+1} = \frac{a_n+2\,b_n}{3} \tag{1.45}$$

$$b_{n+1} = \sqrt[3]{\frac{b_n\,(a_n^2+a_n\,b_n+b_n^2)}{3}} \tag{1.46}$$

$$p_n = \frac{3\,a_n^2}{1-\sum_{k=0}^{n} 3^k\,(a_k^2-a_{k+1}^2)} \tag{1.47}$$

Quintic (5. order) iteration, from the article [13] cf. `src/pi/pi5th.cc`:

$$s_0 = 5(\sqrt{5}-2) \tag{1.48}$$

$$\begin{aligned}
a_0 &= \frac{1}{2} && (1.49)\\
s_{n+1} &= \frac{25}{s_n (z + x/z + 1)^2} \quad \to 1 && (1.50)\\
\text{where } x &= \frac{5}{s_n} - 1 \quad \to 4 && (1.51)\\
\text{and } y &= (x-1)^2 + 7 \quad \to 16 && (1.52)\\
\text{and } z &= \left(\frac{x}{2}\left(y + \sqrt{y^2 - 4x^3}\right)\right)^{1/5} \quad \to 2 && (1.53)\\
a_{n+1} &= s_n^2 a_n - 5^n \left(\frac{s_n^2 - 5}{2} + \sqrt{s_n\,(s_n^2 - 2 s_n + 5)}\right) \quad \to \frac{1}{\pi} && (1.54)\\
a_n - \frac{1}{\pi} &< 16 \cdot 5^n\, e^{-\pi\, 5^n} && (1.55)
\end{aligned}$$

Btw. the 5th order algorithm is the slowest of the above, because compared to the other iterations much more operations are needed for each step.

Cubic (3. order) iteration, from [20], cf. `src/pi/pi3rd.cc`:

$$\begin{aligned}
a_0 &= \frac{1}{3} && (1.56)\\
s_0 &= \frac{\sqrt{3} - 1}{2} && (1.57)\\
r_{k+1} &= \frac{3}{1 + 2\,(1 - s_k^3)^{1/3}} && (1.58)\\
s_{k+1} &= \frac{r_{k+1} - 1}{2} && (1.59)\\
a_{k+1} &= r_{k+1}^2\, a_k - 3^k\,(r_{k+1}^2 - 1) \quad \to \frac{1}{\pi} && (1.60)
\end{aligned}$$

Nonic (9. order) iteration, from [20], cf. `src/pi/pi9th.cc`:

$$\begin{aligned}
a_0 &= \frac{1}{3} && (1.61)\\
r_0 &= \frac{\sqrt{3} - 1}{2} && (1.62)\\
s_0 &= (1 - r_0^3)^{1/3} && (1.63)\\
t &= 1 + 2\, r_k && (1.64)\\
u &= \left(9\, r_k\,(1 + r_k + r_k^2)\right)^{1/3} && (1.65)\\
v &= t^2 + t\,u + u^2 && (1.66)\\
m &= \frac{27\,(1 + s_k + s_k^2)}{v} && (1.67)\\
a_{k+1} &= m\, a_k + 3^{2k-1}\,(1 - m) \quad \to \frac{1}{\pi} && (1.68)\\
s_{k+1} &= \frac{(1 - r_k)^3}{(t + 2\,u)\, v} && (1.69)\\
r_{k+1} &= (1 - s_k^3)^{1/3} && (1.70)
\end{aligned}$$

1.2 Geometric iterations for π

Let r_k and R_k be the radii of a circles that are inscribed and circumscribed, respectively, to a regular polygon with 2^k sides and circumference 2. Then $2\,\pi\,r_k < 2 < 2\,\pi\,R_k$ and the relations

$$r_2 = \frac{1}{4} \tag{1.71}$$

$$R_2 = \frac{1}{\sqrt{8}} \tag{1.72}$$

$$r_{k+1} = \frac{R_k + r_k}{2} \tag{1.73}$$

$$R_{k+1} = \sqrt{R_k\, r_{k+1}} \tag{1.74}$$

allow to compute better and better approximations to π. This is called Cusanus' method, it was discovered around 1450, cf. [44] pp.155-156. Note the different subscripts on the right hand side of the last equation: if they were equal the above iteration would compute the $AGM(r_2, R_2)$.

Archimedes used the circumferences of regular polygons with $3 \cdot 2^k$ that are inscribed (s_k) and circumscribed (t_k) the unit circle.

$$t_0 = 2\sqrt{3} \tag{1.75}$$

$$s_0 = 3 \tag{1.76}$$

$$t_{k+1} = \frac{2\, t_k\, s_k}{t_k + s_k} \tag{1.77}$$

$$s_{k+1} = \sqrt{t_{k+1}\, s_k} \tag{1.78}$$

This is also called the Borchard-Pfaff algorithm. Again, if the subscipts on the right hand side of the last equation were the same then one would compute the harmonic-geometric mean $HGM(a_0, b_0)$ that also has the quadratic convergence property of the AGM. One can identify equation 1.77 as

$$\tan\frac{\phi}{2} = \frac{2\,\tan\phi\,\sin\phi}{\tan\phi + \sin\phi} \tag{1.79}$$

and equation 1.78 as

$$\sin\frac{\phi}{2} = \sqrt{\tan\frac{\phi}{2}\,\sin\phi} \tag{1.80}$$

The quantities in the above algorithms can be identified with (values of) trigonometrical functions and their half-argument relations.

It is easy to contruct one-valued iterations of similar kind. Consider $4 \cdot 2^k$-sided regular polygons circumscribed around the unit circle. The cicumference is used as an approximation for that of the circle: $2\,\pi \approx 8 \cdot 2^k\, x_k$ where x_k is the length of one side of the $4 \cdot 2^k$-gons. One has

$$x_k = \tan\frac{x_0}{2^k} \tag{1.81}$$

Start with the unit square ($x_0 = 1$) and use

$$\tan\frac{\phi}{2} = \frac{\tan\phi}{1 + \sqrt{1 + \tan^2\phi}} \tag{1.82}$$

i.e. iterate

$$x_{k+1} = \frac{x_k}{1 + \sqrt{1 + x_k^2}} \tag{1.83}$$

This can be rewritten in many ways, this one is quite elegant: use

$$\arctan \frac{1}{x} = 2 \arctan \frac{1}{x + \sqrt{x^2 + 1}} \tag{1.84}$$

i.e. iterate

$$\phi_{k+1} = \phi_k + \sqrt{\phi_k^2 + 1} \tag{1.85}$$

The approximation made in these algorithms is always of the type $\sin \phi \approx \phi$ or $\tan \phi \approx \phi$.
Note that all these iterations converge only linear.
Cf. [3] and [45].

1.3 Products for π

Wallis product

$$\begin{aligned} \frac{\pi}{2} &= \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \ldots \\ &= \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1} \end{aligned} \tag{1.86}$$

From

$$\sin 2\,x = 2 \sin x \cos x \tag{1.87}$$

or equivalent

$$\frac{\sin 2\,x}{2\,x} = \frac{\sin x}{x} \cos x \tag{1.88}$$

follows by repeated substitution

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos \frac{x}{2^n} \tag{1.89}$$

using

$$\cos \frac{x}{2} = \sqrt{\frac{1}{2} + \frac{\cos x}{2}} \tag{1.90}$$

$$= \frac{\sqrt{2 + 2 \cos x}}{2} \tag{1.91}$$

and $x = \frac{\pi}{2}$ one gets

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \sqrt{\frac{1}{2}}}} \cdot \ldots \tag{1.92}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdot \ldots \tag{1.93}$$

From

$$\frac{\tan x}{x} = \prod_{n=1}^{\infty} \frac{1}{1 - \tan \frac{x}{2^n}} \tag{1.94}$$

and

$$\cos x = \frac{\sin x}{x} \Big/ \frac{\tan x}{x} \tag{1.95}$$

and formula 1.89 follows

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \tan^2 \frac{x}{2^n}\right)^{2^n - 1} \tag{1.96}$$

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{2\,(k-\frac{1}{2})\,(k+2)}{27\,(k+\frac{2}{3})\,(k+\frac{4}{3})} & 10 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \pi + 6 \\ 0 & 1 \end{bmatrix} \tag{1.97}$$

R.W.Gosper gave

$$\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{5}{2})\,(k+\frac{3}{2})\,(k+3)\,(k+\frac{7}{2})}{64\,(k+\frac{3}{4})\,(k+\frac{5}{4})\,(k+\frac{9}{4})\,(k+\frac{11}{4})} & 48\,(k + \frac{41}{21}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 15\,\pi + 32 \\ 0 & 1 \end{bmatrix} \tag{1.98}$$

and

$$\begin{aligned} &\prod_{k=0}^{\infty} \begin{bmatrix} \frac{(k-\frac{19}{6})\,(k-\frac{13}{6})\,(k+1)\,(k+\frac{19}{6})\,(k+\frac{25}{6})}{64\,(k-\frac{7}{12})\,(k-\frac{1}{12})\,(k+\frac{3}{2})\,(k+\frac{31}{12})\,(k+\frac{37}{12})} & (k - \frac{23}{42})\,(k + \frac{1}{3})\,(k + \frac{11}{6}) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{6175}{2268}\,\pi - \frac{325}{378} & . \\ . & . \end{bmatrix} \end{aligned} \tag{1.99}$$

Gosper gives two matrix products for arctan(x) (and thereby for π): Define

$$K(k,n) := \begin{bmatrix} \frac{k\,x^2}{(n+k)(x^2+1)} & \frac{x}{x^2+1} \\ 0 & 1 \end{bmatrix} \tag{1.100}$$

and

$$N(k,n) := \begin{bmatrix} -\frac{n\,x^2}{n+k} & x \\ 0 & 1 \end{bmatrix} \tag{1.101}$$

then

$$\arctan(x) = \text{upper-right}\left(K(1,\frac{1}{2}) \cdot K(2,\frac{1}{2}) \cdot K(3,\frac{1}{2}) \cdot K(4,\frac{1}{2}) \cdot ...\right) \tag{1.102}$$

$$= \text{upper-right}\left(N(1,\frac{1}{2}) \cdot N(1,\frac{3}{2}) \cdot N(1,\frac{5}{2}) \cdot N(1,\frac{7}{2}) \cdot ...\right) \tag{1.103}$$

1.4 Arctan formulas

Formulas of the form

$$k\,\frac{\pi}{4} = \sum_{i=1}^{N} m_i \arctan \frac{1}{x_i} \tag{1.104}$$

$(k \in \mathbb{N}, \quad m_i \in \mathbb{Z}, \quad x_i \in \mathbb{N})$
where all $x_i^2 + 1$ factor completely into factors in

$$F \quad := \{2, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113\} \tag{1.105}$$

(exceptions are formulas 1.116 and 1.118).

For some of the formulas the factors are given in curly braces[1].

All these formulas were built in 1992 with a mixture of C-programs and Computer Algebra. For the known formulas the original authors are given.

Formulas 1.110 and 1.111 were used for the 100,000 digit computation of π in 1961, see [28]. Machin's formula (1.109) was often used in π-computations before 1960. The computations used the expansion

$$\arctan \frac{1}{x} \quad = \quad \frac{1}{x} - \frac{1}{3\,x^3} + \frac{1}{5\,x^5} - \frac{1}{7\,x^7} + - ... \tag{1.106}$$

Another expansion is given by

$$\arctan \frac{1}{x} \quad = \quad x \left(\frac{1}{x^2+1} + \frac{2}{3} \frac{1}{(x^2+1)^2} + \frac{2 \cdot 4}{3 \cdot 5} \frac{1}{(x^2+1)^3} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \frac{1}{(x^2+1)^4} + ... \right) \tag{1.107}$$

$$\frac{\pi}{4} \quad = \quad \arctan \frac{1}{2} + \arctan \frac{1}{3} \tag{1.108}$$

$\{5\}$ *(Euler, 1706)*

$$\frac{\pi}{4} \quad = \quad 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \tag{1.109}$$

$\{13\}$ *(Machin, 1776)*

$$\frac{\pi}{4} \quad = \quad 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{57} + \arctan \frac{1}{239} \tag{1.110}$$

$\{5,13\}$ *(Størmer, 1896)*

$$\frac{\pi}{4} \quad = \quad 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} \tag{1.111}$$

$\{5,13\}$ *(Gauss)*

$$\frac{\pi}{4} \quad = \quad 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943} \tag{1.112}$$

$\{5,13,61\}$ *(Størmer, 1896)*

$$\begin{aligned} \frac{\pi}{4} \quad = \quad & 88 \arctan \frac{1}{172} + 51 \arctan \frac{1}{239} + 32 \arctan \frac{1}{682} + \\ & +44 \arctan \frac{1}{5357} + 68 \arctan \frac{1}{12943} \end{aligned} \tag{1.113}$$

$\{5,13,61,97\}$ *(Størmer, 1896)*

$$\begin{aligned} \frac{\pi}{4} \quad = \quad & 88 \arctan \frac{1}{192} + 39 \arctan \frac{1}{239} + 100 \arctan \frac{1}{515} - \\ & -32 \arctan \frac{1}{1068} - 56 \arctan \frac{1}{173932} \end{aligned} \tag{1.114}$$

$\{5,13,73,101\}$

[1] the 2 is always omitted.

$$\begin{aligned}\frac{\pi}{4} &= 100 \arctan\frac{1}{319} + 127 \arctan\frac{1}{378} + 71 \arctan\frac{1}{557} - \\ &\quad -15 \arctan\frac{1}{1068} + 66 \arctan\frac{1}{2943} + 44 \arctan\frac{1}{478707}\end{aligned} \tag{1.115}$$

{5,13,17,41,73}

$$\begin{aligned}\frac{\pi}{4} &= 322 \arctan\frac{1}{577} + 76 \arctan\frac{1}{682} + 139 \arctan\frac{1}{1393} + \\ &\quad +156 \arctan\frac{1}{12943} + 132 \arctan\frac{1}{32807} + 44 \arctan\frac{1}{1049433}\end{aligned} \tag{1.116}$$

{5,13,61,89,197}

$$\begin{aligned}\frac{\pi}{4} &= 1074 \arctan\frac{1}{1568} + 657 \arctan\frac{1}{4662} + 183 \arctan\frac{1}{5357} - \\ &\quad -779 \arctan\frac{1}{12943} - 32 \arctan\frac{1}{17923} - 449 \arctan\frac{1}{32807} + \\ &\quad +398 \arctan\frac{1}{390112}\end{aligned} \tag{1.117}$$

{5,13,17,61,89,97}

$$\begin{aligned}\frac{\pi}{4} &= 1587 \arctan\frac{1}{2852} + 295 \arctan\frac{1}{4193} + 593 \arctan\frac{1}{4246} + \\ &\quad +359 \arctan\frac{1}{39307} + 481 \arctan\frac{1}{55603} + 625 \arctan\frac{1}{211050} - \\ &\quad -708 \arctan\frac{1}{390112}\end{aligned} \tag{1.118}$$

{5,13,17,29,97,433}

$$\begin{aligned}\frac{\pi}{4} &= 1074 \arctan\frac{1}{4246} + 1257 \arctan\frac{1}{5357} + 1731 \arctan\frac{1}{6107} + \\ &\quad +295 \arctan\frac{1}{12943} + 625 \arctan\frac{1}{19703} - 481 \arctan\frac{1}{32807} - \\ &\quad -1042 \arctan\frac{1}{39307} + 398 \arctan\frac{1}{390112}\end{aligned} \tag{1.119}$$

{...}

$$\begin{aligned}\frac{\pi}{4} &= 7162 \arctan\frac{1}{12943} + 3796 \arctan\frac{1}{32807} + 2558 \arctan\frac{1}{34208} + \\ &\quad +2729 \arctan\frac{1}{44179} - 708 \arctan\frac{1}{51387} + 2192 \arctan\frac{1}{114669} - \\ &\quad -2805 \arctan\frac{1}{157318} - 3696 \arctan\frac{1}{485298} - 2407 \arctan\frac{1}{24208144}\end{aligned} \tag{1.120}$$

{...}

$$\begin{aligned}\frac{\pi}{4} &= 2805 \arctan\frac{1}{5257} - 398 \arctan\frac{1}{9466} + 1950 \arctan\frac{1}{12943} + \\ &\quad +1850 \arctan\frac{1}{34208} + 2021 \arctan\frac{1}{44179} + 2097 \arctan\frac{1}{85353} + \\ &\quad +1484 \arctan\frac{1}{114669} + 1389 \arctan\frac{1}{330182} + 808 \arctan\frac{1}{485298}\end{aligned} \tag{1.121}$$

{5,13,17,29,37,41,53,61} (Gauss)

$$\begin{aligned}\frac{\pi}{4} &= 50539 \arctan\frac{1}{51387} + 1555 \arctan\frac{1}{114669} \\ &\quad -6601 \arctan\frac{1}{157318} - 20678 \arctan\frac{1}{390112} - 5617 \arctan\frac{1}{485298} \\ &\quad -64126 \arctan\frac{1}{617427} + 10958 \arctan\frac{1}{1984933} - 30569 \arctan\frac{1}{3449051} \\ &\quad +23407 \arctan\frac{1}{22709274} + 25433 \arctan\frac{1}{24208144} \\ &\quad \{\ldots\}\end{aligned} \tag{1.122}$$

$$\begin{aligned}\frac{\pi}{4} &= 36462 \arctan\frac{1}{390112} + 135908 \arctan\frac{1}{485298} \\ &\quad +274509 \arctan\frac{1}{683982} - 39581 \arctan\frac{1}{1984933} \\ &\quad +178477 \arctan\frac{1}{2478328} - 114569 \arctan\frac{1}{3449051} \\ &\quad -146571 \arctan\frac{1}{18975991} + 61914 \arctan\frac{1}{22709274} \\ &\quad -69044 \arctan\frac{1}{24208144} - 89431 \arctan\frac{1}{201229582} \\ &\quad -43938 \arctan\frac{1}{2189376182} \\ &\quad \{5,13,17,29,37,53,61,89,97,101\}\end{aligned} \tag{1.123}$$

$$\begin{aligned}\frac{\pi}{4} &= 446879 \arctan\frac{1}{683982} + 172370 \arctan\frac{1}{1635786} \\ &\quad -193720 \arctan\frac{1}{1984933} + 369078 \arctan\frac{1}{2478328} \\ &\quad +18231 \arctan\frac{1}{3014557} + 21339 \arctan\frac{1}{3449051} \\ &\quad -154139 \arctan\frac{1}{6225244} - 110109 \arctan\frac{1}{18975991} \\ &\quad +80145 \arctan\frac{1}{22709274} - 223183 \arctan\frac{1}{24208144} \\ &\quad -107662 \arctan\frac{1}{201229582} - 216308 \arctan\frac{1}{2189376182} \\ &\quad \{\ldots\}\end{aligned} \tag{1.124}$$

$$\begin{aligned}\frac{\pi}{4} &= 872408 \arctan\frac{1}{1984933} + 619249 \arctan\frac{1}{2298668} \\ &\quad +369078 \arctan\frac{1}{2478328} + 18231 \arctan\frac{1}{3014557} \\ &\quad -1217159 \arctan\frac{1}{5033696} + 911989 \arctan\frac{1}{6225244} \\ &\quad +783649 \arctan\frac{1}{18975991} - 70886 \arctan\frac{1}{22709274} \\ &\quad -374214 \arctan\frac{1}{24208144} - 1044789 \arctan\frac{1}{168623905} \\ &\quad +339217 \arctan\frac{1}{201229582} - 446879 \arctan\frac{1}{284862638} \\ &\quad +402941 \arctan\frac{1}{2189376182} \\ &\quad \{\ldots\}\end{aligned} \tag{1.125}$$

1.5 How to build arctan formulas

For a n-term arctan relation of the form

$$\frac{k\,\pi}{4} \quad = \quad m_1 \arctan\frac{1}{a_1} + m_2 \arctan\frac{1}{a_2} + ... + m_n \arctan\frac{1}{a_n} \tag{1.126}$$

1. Choose a set $F = \{p_1, ..., p_{n-1}\}$ of primes of the form $4k+1$
2. Find $a_1, ...a_y$ so that a_i^2+1 factor completely in $2 \cup F$,
   i.e. $a_i^2 + 1 = 2^{e_{i,0}} \cdot p_1^{e_{i,1}} \cdot p_2^{e_{i,2}} \cdot ... \cdot p_{n-1}^{e_{i,n-1}} \qquad i = 1, 2, ..., y$
   (the factor 2 for the odd x_i is ignored in what follows).
3. If you can't find more than n a_i, i.e. $y < n$, then restart with another F
4. give the $e_{i,j}$ a minus sign if $(a_i \% p_j) > \frac{p_j}{2}$
5. Find the nullspace of the $(n-1) \times y$ matrix $M_{ij} := \{e_{ij}\}$
6. Find linear combinations of the basis vectors of the nullspace that correspond to nice and nontrivial (i.e. $k \neq 0$) arctan relations.

An example: for a 5-term relation

1. Choose $F = \{5, 13, 61, 101\}$
2. Find $\{2, 3, 5, 7, 8, 18, ..., 57, ..., 111, 239, 515, 682, 12943\}$
   (from which i choose the 5 largest $111, 239, 515, 682, 12943$)
3. $y >= 5$ ok.
4. and
5. $M =$

	5	13	61	101
111	0	0	-1	+1
239	0	+4	0	0
515	0	-1	0	2
682	+3	0	+2	0
12943	-4	-3	+1	0

   (e.g. $239^2 + 1 = 2 \cdot 13^4$ and $239\%13 = 5 < 13/2$)
6. we were lucky:

$$\begin{aligned}\frac{1\,\pi}{4} \quad &= \quad 88 \arctan\frac{1}{111} + 7 \arctan\frac{1}{239} - 44 \arctan\frac{1}{515} + \\ &\quad +32 \arctan\frac{1}{682} + 24 \arctan\frac{1}{12943}\end{aligned} \tag{1.127}$$

Often one ends up with a trivial relation, i.e.

$$0 = m_1 \arctan \frac{1}{a_1} + m_2 \arctan \frac{1}{a_2} + ... + m_n \arctan \frac{1}{a_n} \tag{1.128}$$

To build formulas with terms of the type $\arctan \frac{a}{b}$ use the factors of $a^2 + b^2$.

Open questions:

1. What is the upper bound for the a_i for a certain set of factors F ?
2. Is there an algorithm that (in subexponential running time) finds for a certain set $A = \{a_i\}, i = 1, ..., y$ (for which $a_i^2 + 1$ factor completely in F) the arctan relations also for the subsets of A that contain only a_i for which $a_i^2 + 1$ factor completely in a subset of ?
3. Is there a better algorithm to find the a_i for a certain F than these:
   (a) tree search over all products $\prod p_j^{e_{ij}}$
   (checking for each product if the product minus one is square)
   (b) brute force checking all $a = 2...\infty$ if $a^2 + 1$ factors in F

1.6 Ramanujan type formulas

Some nice explicit 'Ramanujan-type' formulas for $1/\pi$ follow. For more formulas and explanation cf. [10] and [12]

The explicit formulas were made in 1994 using Mathematica (for numerical computation of the quantities in the general formulas and for finding the minimal polynomials) and MapleV (for solving the polynomials and numerical verification of the results).

The 'huge' formulas here are given rather for fun than for the computation of pi.

The 'type X' are the same as in [12].

It is

$$(z)_k := z^{\bar{k}} := z(z+1)(z+2)...(z+k-1) \tag{1.129}$$

in what follows.

1.6.1 Type 1 $n = 58$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \left(2\sqrt{2}\,(1103 + 26390\,n)\right) \frac{1}{(99^2)^{2n+1}} \tag{1.130}$$

$$= \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + n\,26390)}{(n!)^4\,396^{4n}} \tag{1.131}$$

(Ramanujan) about 8 correct digits per term

1.6.2 Type 1 $n = 862$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{A + n\,B}{X^{2n+1}} \tag{1.132}$$

(type 1, $n = 862$)

$$\begin{aligned} A \;:=\; & \Big[45219627310440583676349982714551360 35/4 + \\ & +7993776278485234586059121251125632 34\sqrt{2} + \\ & +12\Big(1775012755290923520301237736918207934527539078119087387065649126 1057219 + \\ & +1255123555958829884236839904476079251826408616374387198187634303258534\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{1.133}$$

$$\begin{aligned} B \;:=\; & \Big[961776139508895348591544409130763610 6000 + \\ & +680078430230158868661625397342978215 4400\sqrt{2} \\ & +52003425600\sqrt{2}\Big(342045665867229031517310725375164691366406720471988305929 63 + \\ & +2418628098101856660655230981125577585184945651021683039952 2\sqrt{2}\Big)^{1/2}\Big]^{1/2} \end{aligned} \tag{1.134}$$

$$\begin{aligned} X \;:=\; & 1670141896514232075 + 1180968660568974600\sqrt{2} + \\ & +2736\sqrt{2}\Big(3726272018650177463417915646 03 + \\ & +2634872212933225771559515148 50\sqrt{2}\Big)^{1/2} \end{aligned} \tag{1.135}$$

(each term adds 37 digits)

1.6.3 Type 2 $n = 37$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{2}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^3} \frac{(1123 + 21460\,n)}{4} \frac{1}{882^{2n+1}} \tag{1.136}$$

(Ramanujan)

1.6.4 Type 3a $n = 7 \ldots 163$

$$\begin{aligned} \frac{1}{\pi} &= \frac{1}{\sqrt{-1728\,J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} && (1.137) \\ &= \frac{1}{\sqrt{-1728\,J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^{3n}\,(3n)!\,(n!)^3} \frac{A + n\,B}{J^n} && (1.138) \end{aligned}$$

n	A	B	J	correct digits per term
7	24	189	-125/64	≤ 1
11	60	616	-512/27	1
19	300	4104	-512	3
27	1116	18216	-64000/9	4
43	9468	195048	-80^3	6
67	122124	3140242	-440^3	8
163	163096908	6541681608	-53360^3	15

The last ($n = 163$) is known as Chudnovsky's formula:

$$\frac{1}{\pi} = \frac{6541681608}{\sqrt{640320}^3} \sum_{k=0}^{\infty} \left(\frac{13591409}{545140134} + k \right) \left(\frac{(6k)!}{(k!)^3\,(3k)!} \frac{(-1)^k}{640320^{3k}} \right) \tag{1.139}$$

$$= \frac{12}{\sqrt{640320}^3} \sum_{k=0}^{\infty} (-1)^k \frac{(6k)!}{(k!)^3\,(3k)!} \frac{13591409 + k\,545140134}{(640320)^{3k}} \tag{1.140}$$

1.6.5 Type 3c $n = 1555$

$$\frac{1}{\pi} = \frac{1}{\sqrt{-12\,J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{1.141}$$

$$= \frac{1}{\sqrt{-12\,J}} \sum_{n=0}^{\infty} \frac{(6n)!}{12^n\,(3n)!\,(n!)^3} \frac{A + n\,B}{J^n} \tag{1.142}$$

(type 3c, $n = 1555$)

$$\begin{aligned} A := {} & 5280419026080999965452185 + 2361475178400070170568800\,\sqrt{5} + \\ & +32\,\sqrt{5}\big(10891728551171178200467436212395209160385656017 + \\ & +4870929086578810225077338534541688721351255040\,\sqrt{5}\big)^{1/2} \end{aligned} \tag{1.143}$$

$$\begin{aligned} B := {} & 654159204458052267524145750 + 292548889855077669080467200\,\sqrt{5} + \\ & +209664\,\sqrt{3110}\Big(6260208323789001636993322654444020882161 + \\ & +2799650273060444296577206890718825190235\,\sqrt{5}\Big)^{1/2} \end{aligned} \tag{1.144}$$

$$\begin{aligned} J := {} & -\Big[17897749588626020 + 8004116944887336\,\sqrt{5} + \\ & +108\,\sqrt{5}\big(10985234579463550323713318473 + \\ & +4912746253692362754607395912\,\sqrt{5}\big)^{1/2}\Big]^3 \end{aligned} \tag{1.145}$$

(each term adds 50 correct digits)

$$\begin{aligned} 0 = {} & 91056965337194438815073158624225 + \\ & +9214187265360390391808927003100\,A + \\ & +672035320036821921804675631270\,A^2 - \end{aligned} \tag{1.146}$$

$$-2112167610432399986180874 0\,A^3 + A^4$$

$$\begin{aligned} 0 &= 175141800181373873265651313890457956000 + \\ &\quad +193756947585743300725193322013380000\,B + \\ &\quad +2063419786805410130433556462222680\,B^2 - \\ &\quad -26166368178322090700096583000\,B^3 + B^4 \end{aligned} \tag{1.147}$$

$$\begin{aligned} 0 &= 27192565672854630400 + \\ &\quad +49698245345181030400\,U + \\ &\quad +2288545308972778272 0\,U^2 - \\ &\quad -71590998354504080\,U^3 + U^4 \end{aligned} \tag{1.148}$$

$$J = -U^3 \tag{1.149}$$

1.6.6 Type 3b $n = 190$

$$\frac{1}{\pi} = \frac{1}{\sqrt{3J}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{3}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!^3} \frac{A + n\,B}{J^n} \tag{1.150}$$

(type 3b, $n = 190$)

$$\begin{aligned} A &:= 21242668516504965 + \\ &\quad +15020834958518500\sqrt{2} + \\ &\quad +2\sqrt{5}\Big(4512509642758656825164561014165 9 + \\ &\quad +3190826168564331290217358543425 0\sqrt{2}\Big)^{1/2} \end{aligned} \tag{1.151}$$

$$\begin{aligned} B &:= 1839779353703421900 + \\ &\quad +1300920456890691000\sqrt{2} + \\ &\quad +24337404\sqrt{10}\Big(11429124767130244966 67 + \\ &\quad +808161162586491705750\sqrt{2}\Big)^{1/2} \end{aligned} \tag{1.152}$$

$$\begin{aligned} J &:= \Big[71864175655 + 22725423252\sqrt{10} + \\ &\quad +2808\sqrt{5}\left(261993316778681 + 82849561276216\sqrt{10}\right)^{1/2}\Big]^3 \end{aligned} \tag{1.153}$$

(each term adds 34 digits)

$$\begin{aligned} 0 &= 18983882886895192207942622025 - \\ &\quad -22331544571858356551863737 00\,A + \\ &\quad +570499890229502944324099 0\,A^2 - \\ &\quad 84970674066019860\,A^3 + A^4 \end{aligned} \tag{1.154}$$

$$\begin{aligned} 0 &= 11316047287507303785105891917318400- \\ &-14356404679179043063243923292800 0\,B+ \\ &+21396235898865291113024998560\,B^2- \\ &-7359117414813687600\,B^3+B^4 \end{aligned} \tag{1.155}$$

$$\begin{aligned} 0 &= 1860185517864501025-1262383694834359900\,U+ \\ &+44498697145120230\,U^2-287456702620\,U^3+U^4 \end{aligned} \tag{1.156}$$

$$J = -U^3 \tag{1.157}$$

How to build Ramanujan type formulas:

1. Read the definitions of the general formulas.
2. Pick out the necessary definitions from Borweins' book ([10]).
3. Write a mupad-package that implements all the needed quantities (steal from the mathematica package `src/pi/bucket/piram.m`).
4. For each formula and n do
   (a) Get numeric approximations for the quantities you need (e.g. f_n and J_n), compute 500 digits or so.
   (b) Find the minimal polynomials for those quantities.
   (c) Solve the polynomials & beautify the results.
   (d) Check the symbolic results by comparing them to the quantities they were made of, using a higher precision than before.

For more formulas and explanation cf. [12] and [10].

1.7 Approximations for π

In what follows $\pi(n)$ denotes an $n-$digit approximation for π.

$$\pi(2) = \frac{22}{7} = 3.1428... \tag{1.158}$$

$$\pi(6) = \frac{355}{113} = 3.14159292... \tag{1.159}$$

The last approximation tells us that it is a particularly bad idea to use π as an irrational value e.g. in chaos theoretic programs: it is almost up to floating point (single-) precision a rational number with pretty small denominator. Use $\phi = \frac{\sqrt{5}-1}{2}$ instead:

$$\pi(2) = \sqrt{3}+\sqrt{2} = 3.1462... \tag{1.160}$$

Ramanujan [23] gives in his paper (among many other):

$$\pi(5) = \frac{12}{\sqrt{22}} \log\left(\sqrt{2}+2\right) \tag{1.161}$$

$$\pi(15) = \frac{12}{\sqrt{130}} \log\left(\frac{1}{2}\sqrt{2}\left(\sqrt{5}+2\right)\left(\sqrt{13}+3\right)\right) \tag{1.162}$$

$$\pi(16) = \frac{24}{\sqrt{142}} \log\left(\frac{1}{2}\sqrt{7\sqrt{2}+10}+\sqrt{11\sqrt{2}+10}\right) \tag{1.163}$$

$$\pi(18) = \frac{12}{\sqrt{190}} \log\left(\left(\sqrt{10}+3\right)\left(2\sqrt{2}+\sqrt{10}\right)\right) \tag{1.164}$$

$$\pi(22) = \frac{12}{\sqrt{310}} \log\left(\left(\sqrt{2}+2\right)\left(\sqrt{5}+3\right)\left(2\sqrt{10}+\sqrt{20\sqrt{10}+61}+5\right)\right) \tag{1.165}$$

$$\pi(31) = \frac{4}{\sqrt{522}} \log\left(\frac{1}{256}\sqrt{2}\left(\sqrt{29}+5\right)^3\left(11\sqrt{6}+5\sqrt{29}\right)\left(\sqrt{3\sqrt{6}+5}+\sqrt{3}\sqrt{\sqrt{6}+3}\right)^6\right) \tag{1.166}$$

From the definition of the J-function

$$\begin{aligned} J(\tau) \;:=\; & 1/q + 744 + 196884\,q + 21493760\,q^2 + 864299970\,q^3 + 20245856256\,q^4 + \\ + \; & 333202640600\,q^5 + 4252023300096\,q^6 + 44656994071935\,q^7 + 401490886656000\,q^8 + \\ + \; & 3176440229784420\,q^9 + 22567393309593600\,q^{10} + \ldots \end{aligned} \tag{1.167}$$

(where $q := e^{2\,i\,\pi\,\tau}$) it is possible to give approximations to π for certain values of τ (cf. [8]).
E.g.

$$\pi(17) = \frac{\log\left(5280^3+744\right)}{\sqrt{67}} \tag{1.168}$$

$$\pi(30) = \frac{\log\left(640320^3+744\right)}{\sqrt{163}} \tag{1.169}$$

for $\tau = 67$ and $\tau = 163$, respectively.

As q is close to an integer for these values, one can take more terms from the above series to get (approximating q by $[q]$) better approximations of the same type. With $z := -J(\frac{1+i\sqrt{163}}{2}) + 744 = 640320^3 + 744$ one gets

$$\pi(46) = \frac{\log\left(z - 196884/z\right)}{\sqrt{163}} \tag{1.170}$$

$$\pi(60) = \frac{\log\left(z - 196884/z + 21493760/z^2\right)}{\sqrt{163}} \tag{1.171}$$

Using more terms doesn't improve the accuracy anymore because of the approximation made for q.

1.8 Continued fractions for π

1.8.1 The simple continued fraction for π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}} \tag{1.172}$$

there is no pattern in the occuring numbers,

1.8.2 other continued fractions for π

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cfrac{11^2}{2 + ...}}}}}} \tag{1.173}$$

(Brouncker, 1658)

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \cfrac{5^2}{11 + ...}}}}} \tag{1.174}$$

(cf. [43])

$$\frac{6}{\pi^2 - 6} = 1 + \cfrac{1^2}{1 + \cfrac{1 \cdot 2}{1 + \cfrac{2^2}{1 + \cfrac{2 \cdot 3}{1 + \cfrac{3^2}{1 + \cfrac{3 \cdot 4}{1 + \cfrac{4^2}{1 + ...}}}}}}} \tag{1.175}$$

$$\frac{12}{\pi^2} = 1 + \cfrac{1^4}{3 + \cfrac{2^4}{5 + \cfrac{3^4}{7 + \cfrac{4^4}{9 + \cfrac{5^4}{+...}}}}} \tag{1.176}$$

1.9 Series for π

$$\frac{\pi}{4} = \arctan 1 \tag{1.177}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} + -... \tag{1.178}$$

(Gregory, 1671). The error of the truncated series is about one half of the first neglected term.

Applying the Euler transform (cf. [41] p.253-255) to 1.177 gives the series

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + ... \tag{1.179}$$

$$= 1 + \frac{1}{3}\left(1 + \frac{2}{5}\left(1 + \frac{3}{7}\left(1 + ...\right)\right)\right) \tag{1.180}$$

which is used in the spigot algorithm, cf. [9].

In [1] the following acceleration of the arctan-series is given:

$$\arctan x = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{2k+1} + (-1)^n \frac{1}{2} x^{2n-1} \sum_{k=0}^{\infty} \left(\frac{2x^2}{x^2+1}\right)^{k+1} \frac{k!}{((2n+1))_{k+1}} \tag{1.181}$$

where $((2n+1))_{k+1}$ denotes $(2n+1)(2n+3)...(2n+2k+1)$. The error of the truncated series is less than $|1+x|$ times the first neglected term.

For $x = 1$ (formula 1.177) and $n = 500$ this is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - +... - \frac{1}{999} + \tag{1.182}$$

$$+ \frac{1}{2}\left(\frac{1}{1001} + \frac{1!}{1001 \cdot 1003} + \frac{2!}{1001 \cdot 1003 \cdot 1005} + \frac{3!}{1001 \cdot 1003 \cdot 1005 \cdot 1007} + ...\right)$$

$$\frac{\pi}{6} = \arcsin \frac{1}{2} \tag{1.183}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3 \cdot 2^3}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{5 \cdot 2^5}\right) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{7 \cdot 2^7}\right) + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\left(\frac{1}{9 \cdot 2^9}\right) + ... \tag{1.184}$$

(Newton, 1665)

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} - \frac{1}{8 \cdot 9 \cdot 10} + \frac{1}{10 \cdot 11 \cdot 12} - +... \tag{1.185}$$

$$\frac{4}{\pi} = 1 + \frac{1}{4} + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^3 + \left(\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\right)^4 + ... \tag{1.186}$$

(Gauss)

$$\frac{4}{\pi} = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \frac{1}{2}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + ... \tag{1.187}$$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - 13\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^3 + -... \tag{1.188}$$

(Ramanujan, cf. p.7 in [46])

$$\pi = 3\sqrt{3} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!\, n} \tag{1.189}$$

$$\pi = 3\sqrt{3}\sum_{n=0}^{\infty} \frac{1}{(3n+1)\,(3n+2)} \tag{1.190}$$

Using the identity

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{1.191}$$

it is possible to compute some hexadecimal digits of π *without* computing any of the preceding digits. See the article [35] for the algorithm.

A similar series is

$$\pi = \sum_{i=0}^{\infty} \frac{(-1)^i}{4^i} \left(\frac{2}{4i+1} + \frac{2}{4i+2} + \frac{1}{4i+3} \right) \tag{1.192}$$

$$\pi = 3 + \frac{1}{60}\left(8 + \frac{2\cdot 3}{7\cdot 8\cdot 3}\left(13 + \frac{3\cdot 5}{10\cdot 11\cdot 3}\left(18 + \frac{4\cdot 7}{13\cdot 14\cdot 3}\,(23+\ldots)\right)\right)\right) \tag{1.193}$$

(Gosper, cf. [52])

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sum_{k=1}^{n} k^2} = 6\,(\pi - 3) \tag{1.194}$$

(from [10], p.101)

F.Bellard gives

$$740025\,\pi + 20379280 = \sum_{n=1}^{\infty} \frac{3\,P(n)}{\binom{7\,n}{2\,n}\;2^{n-1}} \tag{1.195}$$

where

$$\begin{aligned} P(n) \;:=\; & -885673181\,n^5 + 3125347237\,n^4 - 2942969225\,n^3 \\ & +1031962795\,n^2 - 196882274\,n + 10996648 \end{aligned} \tag{1.196}$$

$$\begin{aligned} \pi &= \log\frac{1}{x} - 2\,x^4 - 13\,x^8 - \frac{368}{3}\,x^{12} - \ldots \\ & \quad \text{where} \qquad x = \frac{1}{2}\,\frac{2^{1/4}-1}{2^{1/4}+1} \end{aligned} \tag{1.197}$$

1.10 Miscellaneous formulas for π

$$e^{\sqrt{-1}\,\pi} = -1 \tag{1.198}$$

(Euler)

In [2] integrals of the form

$$I_{n,m} = \int_0^1 \frac{x^m\,(1-x)^n}{(1+x^2)} \tag{1.199}$$

examples are

$$
\begin{aligned}
I_{4,4} &= \int_0^1 \frac{x^4\,(1-x)^4}{(1+x^2)} = \frac{22}{7} - \pi && (1.200)\\
I_{2,4} &= \pi - \frac{47}{15} && (1.201)\\
I_{6,12} &= 16\,\pi - \frac{153966181}{3063060} && (1.202)\\
I_{32,32} &= 16384\,\pi - \frac{31694514838868667276634759966 4}{61576400213688659766216 75} && (1.203)
\end{aligned}
$$

The fractions on the rhs. are, as shown in the paper, approximations for π, the last gives 23 correct digits.

$$
\pi = \frac{426880\,\sqrt{10005}}{13591409\,\left({}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;A) - B\,{}_3F_2(\frac{7}{6},\frac{3}{2},\frac{11}{6};2,2;A)\right)} \qquad (1.204)
$$

where

$$
\begin{aligned}
A &:= -\frac{1}{151931373056000} && (1.205)\\
& && (1.206)\\
B &:= \frac{30285563}{1651969144908540723200} && (1.207)
\end{aligned}
$$

(Chudnovskys)

From the Poisson summation formula follows:

$$
\begin{aligned}
\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right) &= \pi && (1.208)\\
\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 &= \pi && (1.209)\\
\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^3 &= \frac{4}{3}\,\pi && (1.210)\\
\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^4 &= \frac{2}{3}\,\pi && (1.211)
\end{aligned}
$$

(cf. appendix B).

1.11 A bit recursion for $1/\pi$

$$
\begin{aligned}
a_0 &:= \tan(1) && (1.212)\\
a_{k+1} &:= \frac{2\,a_k}{1-a_k^2} && (1.213)\\
b(x) &:= \begin{cases} 1 & \text{if} \quad x < 0 \\ 0 & \text{else} \end{cases} && (1.214)
\end{aligned}
$$

then

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{1}{\pi} \tag{1.215}$$

$$\sum_{k=0}^{\infty} \frac{b(a_k)}{2^{k+1}} = \frac{\arctan(a_0)}{\pi} \tag{1.216}$$

See and the paper [40].

1.12 A self correcting iteration for π

Use

$$\sin \frac{\pi}{2} = 0 \tag{1.217}$$

therefor the iteration defined by

$$x_0 = \frac{\pi}{2} + \epsilon_0 \tag{1.218}$$

$$x_{k+1} = x_k + \sin x_k \tag{1.219}$$

converges towards π.
Convergence is of third order: if $x_k = \frac{\pi}{2} + \epsilon_k$ then

$$\epsilon_{k+1} \approx \frac{\epsilon_k^3}{6} \tag{1.220}$$

Of course, this is not an 'efficient' iteration as the computation of a sine function is required. Similar iterations exist for cos() and tan(), see [42].

Appendix A

More arctan formulas

In what follows let $\mathbf{A}\frac{1}{x}$ denote $\arctan\frac{1}{x}$.

$$\begin{aligned}
\arctan\frac{1}{x} &= \arctan\frac{1}{x+d} + \arctan\frac{1}{x+\frac{x^2+1}{d}} && \text{(A.1)}\\
\arctan\frac{1}{x} + \arctan\frac{1}{y} &= \arctan\frac{1}{\frac{x\,y-1}{x+y}} && \text{(A.2)}
\end{aligned}$$

$$\begin{aligned}
\arctan\frac{1}{z} &= \arctan\frac{1}{x_1} + \arctan\frac{1}{x_2} + \ldots + \arctan\frac{1}{x_n} && \text{(A.3)}\\
\Longleftrightarrow\quad & z = \frac{Re(\prod_{k=1}^{n}(x_k+i))}{Im(\prod_{k=1}^{n}(x_k+i))} && \text{(A.4)}\\
\Longleftrightarrow\quad & (z^2+1) = \frac{\prod_{k=1}^{n}(x_k^2+1)}{\left(Im(\prod_{k=1}^{n}(x_k+i))\right)^2} && \text{(A.5)}
\end{aligned}$$

$$\begin{aligned}
\arctan\frac{1}{x} &= \arctan\frac{1}{y} + \arctan\frac{1}{z} && \text{(A.6)}\\
\Longleftrightarrow\quad & (x^2+1) = \frac{(y^2+1)(z^2+1)}{(y+z)^2} = (y-x)(z-x) && \text{(A.7)}
\end{aligned}$$

$$\begin{aligned}
\frac{k\,\pi}{4} &= \arctan\frac{b_1}{a_1} + \arctan\frac{b_2}{a_2} + \ldots + \arctan\frac{b_n}{a_n} && \text{(A.8)}\\
\Longleftrightarrow\quad & (a_1+i\,b_1)(a_2+i\,b_2)\ldots(a_n+i\,b_n)(1-i)^k \in \mathbb{R} && \text{(A.9)}
\end{aligned}$$

$$\begin{aligned}
\frac{k\,\pi}{4} &= \arctan\frac{1}{a_1} + \arctan\frac{1}{a_2} + \ldots + \arctan\frac{1}{a_n} && \text{(A.10)}\\
\Longleftrightarrow\quad & (a_1+i)(a_2+i)\ldots(a_n+i)(1-i)^k \in \mathbb{R} && \text{(A.11)}
\end{aligned}$$

$$\begin{aligned}
\frac{k\,\pi}{4} &= m_1\,\arctan\frac{1}{a_1} + m_2\,\arctan\frac{1}{a_2} + \ldots + m_n\,\arctan\frac{1}{a_n} && \text{(A.12)}\\
\Longleftrightarrow\quad & (a_1+i)^{m_1}(a_2+i)^{m_2}\ldots(a_n+i)^{m_n}(1-i)^k \in \mathbb{R} && \text{(A.13)}
\end{aligned}$$

Let

$$a > b, \qquad gcd(a,b) = 1, \quad a\,\alpha + b\,\beta = 1 \tag{A.14}$$

then

$$\mathbf{A}\frac{1}{a}b \quad = \quad \mathbf{A}\frac{1}{-\beta}\alpha + \arctan\frac{1}{a\,\beta - b\,\alpha} \tag{A.15}$$

because if

$$r \quad := \quad a\,\beta - b\,\alpha \tag{A.16}$$

$$\arctan\frac{1}{z} \quad := \quad \arctan\frac{1}{a/b} + \arctan\frac{1}{r} \tag{A.17}$$

then

$$z \quad = \quad \frac{(a/b)\,r - 1}{(a/b) + r} \tag{A.18}$$

$$= \quad \frac{a(\alpha\,b - \beta\,a) - b}{a + b(\alpha\,b - \beta\,a)} \tag{A.19}$$

$$= \quad \frac{a(\alpha\,b - \beta\,a) - b(1)}{a(1) + b(\alpha\,b - \beta\,a)} \tag{A.20}$$

$$= \quad \frac{a(\alpha\,b - \beta\,a) - b(\alpha\,a + \beta\,b)}{a(\alpha\,a + \beta\,b) + b(\alpha\,b - \beta\,a)} \tag{A.21}$$

$$= \quad \frac{-a^2\,\beta - b^2\,\beta}{\alpha\,a^2 + \alpha\,b^2} \tag{A.22}$$

$$= \quad \frac{-\beta}{\alpha} \tag{A.23}$$

$$\mathbf{A}\frac{1}{x} \quad = \quad \mathbf{A}\frac{1}{x+1} + \mathbf{A}\frac{1}{x^2+x+1} \tag{A.24}$$

$$\frac{\pi}{4} \quad = \quad \mathbf{A}\frac{1}{1} = \sum_{x=1}^{\infty}\mathbf{A}\frac{1}{x^2+x+1} = \tag{A.25}$$

$$= \quad \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} + \mathbf{A}\frac{1}{13} + \mathbf{A}\frac{1}{21} + \mathbf{A}\frac{1}{31} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{57} + ... \tag{A.26}$$

$$\mathbf{A}\frac{1}{2x+1} \quad = \quad \mathbf{A}\frac{1}{2x+3} + \mathbf{A}\frac{1}{2x^2+4x+2} \tag{A.27}$$

$$\frac{\pi}{4} \quad = \quad \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2n^2} \tag{A.28}$$

$$= \quad \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{8} + \mathbf{A}\frac{1}{18} + \mathbf{A}\frac{1}{50} + \mathbf{A}\frac{1}{98} + \mathbf{A}\frac{1}{128} + \dots \tag{A.29}$$

define

$$jt(a_0, a_1, m_1, m_2, 0) := a0 \quad (A.30)$$
$$jt(a_0, a_1, m_1, m_2, 1) := a1 \quad (A.31)$$
$$jt(a_0, a_1, m_1, m_2, n) := m_1\, jt(a_0, a_1, m_1, m_2, n-1) + m_2\, jt(a_0, a_1, m_1, m_2, n-2) \quad (A.32)$$

then

$$jt(a_0, a_1, m_1, m_2, n) = C_1\, x_1^n + C_2\, x_2^n \qquad \text{where:} \quad (A.33)$$
$$x_{1,2}^2 = m_1\, x_{1,2} + m_2, \quad C_1 + C_2 = a_0, \quad C_1\, x_1 + C_2\, x_2 = a_1 \quad (A.34)$$

$$\frac{\pi}{8} = \sum_{n=1}^{\infty} (-1)^n \mathbf{A} \frac{1}{jt(0,1,2,1,2n)} = \mathbf{A}\frac{1}{2} - \mathbf{A}\frac{1}{12} + \mathbf{A}\frac{1}{70} - \mathbf{A}\frac{1}{408} + \mathbf{A}\frac{1}{2378} - + \dots \quad (A.35)$$
$$\frac{\pi}{8} = \sum_{n=1}^{\infty} \mathbf{A} \frac{1}{jt(1,1,2,1,2n)} = \quad (A.36)$$
$$= \sum_{n=1}^{\infty} (-1)^n\, \mathbf{A} \frac{1}{(1-\sqrt{2})^{2n} + (1+\sqrt{2})^{2n}} \quad (A.37)$$
$$= \mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{99} + \mathbf{A}\frac{1}{577} + \mathbf{A}\frac{1}{3363} + \dots \quad (A.38)$$

$$f(n) := jt(0,1,2,1,2n)\, jt(0,1,2,1,2n+1)^2 + jt(1,1,2,1,2n+1) \quad (A.39)$$
$$\text{eg.} \quad f(0) = 2 \cdot 5^2 + 7 = 57, \quad f(1) = 12 \cdot 29^2 + 41 = 10133, \quad \dots \quad (A.40)$$
$$\frac{\pi}{8} = \sum_{n=1}^{\infty} 3^n\, \mathbf{A} \frac{1}{jt(0,1,2,1,2n+1)} - \sum_{n=1}^{\infty} 3^n\, \mathbf{A} \frac{1}{f(n)} \quad (A.41)$$
$$= \mathbf{A}\frac{1}{5} + 3\left(\mathbf{A}\frac{1}{29} + 3\left(\mathbf{A}\frac{1}{169} + 3\left(\mathbf{A}\frac{1}{5741} + \dots\right)\right)\right) - \quad (A.42)$$
$$- \left(\mathbf{A}\frac{1}{57} + 3\left(\mathbf{A}\frac{1}{10133} + 3\left(\mathbf{A}\frac{1}{1999509} + \dots\right)\right)\right)$$
$$\mathbf{A}\frac{1}{P_n} = 4\,\mathbf{A}\frac{1}{Q_{n+2}} + \mathbf{A}\frac{1}{P_{n+4}} - 2\,\mathbf{A}\frac{1}{q(Q_{n+2})} (\text{n odd}) \quad (A.43)$$
$$\frac{\pi}{4} = \mathbf{A}\frac{1}{P_1} = 4\left(\mathbf{A}\frac{1}{5} + \mathbf{A}\frac{1}{169} + \mathbf{A}\frac{1}{5741} + \dots\right) - \quad (A.44)$$
$$- 2\left(\mathbf{A}\frac{1}{q(5)} + \mathbf{A}\frac{1}{q(169)} + \mathbf{A}\frac{1}{q(5741)} + \dots\right)$$

$$\text{with } g(n) := jt(0,1,2,1,n+3)\, jt(0,1,2,1,n+2)^2 + jt(1,1,2,1,n+2) \quad (A.45)$$
$$\mathbf{A}\frac{1}{jt(1,1,2,1,n)} = 5\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+2)} + 2\,\mathbf{A}\frac{1}{jt(1,1,2,1,n+3)} \quad (A.46)$$
$$- \mathbf{A}\frac{1}{q(jt(0,1,2,1,n+2))} + 2\,\mathbf{A}\frac{1}{g(n)} \quad (\text{n odd})$$
$$\frac{\pi}{4} = 2\,\mathbf{A}\frac{1}{3} + \mathbf{A}\frac{1}{7} = 2\,\mathbf{A}\frac{1}{3} + 2\,\mathbf{A}\frac{1}{17} + \mathbf{A}\frac{1}{41} = \quad (A.47)$$

$$= \left(12\,\mathbf{A}\frac{1}{17} + 5\,\mathbf{A}\frac{1}{41}\right) - 2\,\mathbf{A}\frac{1}{70} - 4\,\mathbf{A}\frac{1}{307} \tag{A.48}$$

$$= \left(70\,\mathbf{A}\frac{1}{99} + 29\,\mathbf{A}\frac{1}{239}\right) - 2\,\mathbf{A}\frac{1}{70} - 4\,\mathbf{A}\frac{1}{307} - 12\,\mathbf{A}\frac{1}{12238} - 24\,\mathbf{A}\frac{1}{58911} \tag{A.49}$$

$$= \left(408\,\mathbf{A}\frac{1}{577} + 169\,\mathbf{A}\frac{1}{1393}\right) - \tag{A.50}$$

$$- \;(2\,\mathbf{A}\frac{1}{70} + 4\,\mathbf{A}\frac{1}{307} + 12\,\mathbf{A}\frac{1}{12238} + 24\,\mathbf{A}\frac{1}{58911} + 70\,\mathbf{A}\frac{1}{...} + 140\,\mathbf{A}\frac{1}{...})$$

$$\frac{\pi}{4} = 2\,(\sqrt{2}-1) - \sum_{n=1}^{\infty}\left(jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{q(jt(0,1,2,1,2n+1))} + \right. \tag{A.51}$$

$$\left. + \;2\,jt(0,1,2,1,2n)\,\mathbf{A}\frac{1}{g(2n+1)}\right)$$

$$\frac{\pi}{4} = 3\sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)} - \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{jt(1,1,3,1,2n)} \tag{A.52}$$

$$= 3\left(\mathbf{A}\frac{1}{4} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{469} + \mathbf{A}\frac{1}{5116} + \ldots\right) \tag{A.53}$$

$$- \left(\mathbf{A}\frac{1}{q(4)} + \mathbf{A}\frac{1}{q(43)} + \mathbf{A}\frac{1}{q(469)} + \mathbf{A}\frac{1}{q(5116)} + \ldots\right)$$

$$= 3\left(\mathbf{A}\frac{1}{4} + \mathbf{A}\frac{1}{43} + \mathbf{A}\frac{1}{469} + \mathbf{A}\frac{1}{5116} + \ldots\right) \tag{A.54}$$

$$- \left(\mathbf{A}\frac{1}{38} + \mathbf{A}\frac{1}{39818} + \mathbf{A}\frac{1}{51581558} + \ldots\right)$$

$$q(n) := \frac{3n+n^3}{2} \tag{A.55}$$

$$F_n := jt(0,1,1,1,n) \tag{A.56}$$

$$= Fibonacci(n) = 1,1,2,3,5,8,13,21,34,55,... \tag{A.57}$$

$$\mathbf{A}\frac{1}{x} = 2\,\mathbf{A}\frac{1}{2x} - \mathbf{A}\frac{1}{3x+4x^3} \tag{A.58}$$

$$\frac{\pi}{4} = 1 - \sum_{n=0}^{\infty}2^n\,\mathbf{A}\frac{1}{3\,2^n + 4\,2^{3n}} = 1 - \sum_{n=0}^{\infty}2^n\,\mathbf{A}\frac{1}{q(2^{n+1})} \tag{A.59}$$

$$= 1 - \left(\mathbf{A}\frac{1}{7} + 2\,\mathbf{A}\frac{1}{38} + 4\,\mathbf{A}\frac{1}{268} + 8\,\mathbf{A}\frac{1}{2072} + 16\,\mathbf{A}\frac{1}{16432} + \ldots\right) \tag{A.60}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{F_{2n+1}} \tag{A.61}$$

$$= \mathbf{A}\frac{1}{2} + \mathbf{A}\frac{1}{5} + \mathbf{A}\frac{1}{13} + \mathbf{A}\frac{1}{34} + \mathbf{A}\frac{1}{89} + \mathbf{A}\frac{1}{233} + \ldots$$

$$\mathbf{A}\frac{1}{F_{2n}} = 3\,\mathbf{A}\frac{1}{F_{2n+2}} - \mathbf{A}\frac{1}{F_{2n+4}} - \mathbf{A}\frac{1}{q(F_{2n+2})} \Rightarrow \tag{A.62}$$

$$\frac{\pi}{4} = \frac{3\sqrt{5}-5}{2} - \sum_{n=1}^{\infty}F_{2n}\,\mathbf{A}\frac{1}{q(F_{2n+2})} \tag{A.63}$$

$$= \frac{3\sqrt{5}-5}{2} - \left(1\,\mathbf{A}\frac{1}{18} + 3\,\mathbf{A}\frac{1}{268} + 8\,\mathbf{A}\frac{1}{4662} + 21\,\mathbf{A}\frac{1}{83270} + \ldots\right) \tag{A.64}$$

$$\mathbf{A}\frac{1}{x+\sqrt{x^2-4}} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{x\,jt(0,1,x,-1,n)^2} \tag{A.65}$$

for $x=2$:

$$\mathbf{A}\frac{1}{2+\sqrt{2^2-4}} = \mathbf{A}\frac{1}{1} \tag{A.66}$$

$$= \frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2\,n^2} \tag{A.67}$$

for $x=4$:

$$\mathbf{A}\frac{1}{4+\sqrt{4^2-4}} = \mathbf{A}\frac{1}{2+\sqrt{3}} \tag{A.68}$$

$$= \frac{\pi}{12} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{4\,jt(0,1,4,-1,n)^2} \tag{A.69}$$

$$= \mathbf{A}\frac{1}{4}+\mathbf{A}\frac{1}{64}+\mathbf{A}\frac{1}{900}+\mathbf{A}\frac{1}{12544}+\mathbf{A}\frac{1}{174724}+\mathbf{A}\frac{1}{2433600}+\ldots \tag{A.70}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2-2x+1}-\mathbf{A}\frac{1}{4x^2+2x+1} \Rightarrow \tag{A.71}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2-2n+1}+\mathbf{A}\frac{1}{4n^2+2n+1}\right] \tag{A.72}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2+2x+1}-\mathbf{A}\frac{1}{2x^3+x}+\mathbf{A}\frac{1}{4x^3+3x} \tag{A.73}$$

$$\mathbf{A}\frac{1}{2x^2} = 2\,\mathbf{A}\frac{1}{4x^2-2x+1}+\mathbf{A}\frac{1}{2x^3+x}-\mathbf{A}\frac{1}{4x^3+3x} \Rightarrow \tag{A.74}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\left[\mathbf{A}\frac{1}{4n^2+2n+1}+\mathbf{A}\frac{1}{2n^3+n}-\mathbf{A}\frac{1}{4n^3+3n}\right] \tag{A.75}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty}\mathbf{A}\frac{1}{2\,jt(1,3,4,-1,n)^2} \tag{A.76}$$

$$= \mathbf{A}\frac{1}{2}+\mathbf{A}\frac{1}{18}+\mathbf{A}\frac{1}{242}+\mathbf{A}\frac{1}{3362}+\mathbf{A}\frac{1}{46818}+\mathbf{A}\frac{1}{652082}+\ldots \tag{A.77}$$

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\mathbf{A}\frac{1}{2n^2} = \text{(by changing the order of summation)} \tag{A.78}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)2^{2n+1}}\zeta(4n+2) \tag{A.79}$$

Appendix B

Sums and integrals of $sin(x)/x$

For $k \in \mathbb{Z}$

$$\sum_{x=-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^k = \int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^k dx \tag{B.1}$$

$$= r_k \, \pi \tag{B.2}$$

where r_k is a rational number. The equality of the integral with the sum follows from the Euler-MacLaurin summation formula (cf. [6] section 9.5 and [7] section 0.7.1.3). That the sums are equal to π times a rational follows from the Poisson summation formula and that

$$\frac{sin(x)}{x} = \int_{-\infty}^{+\infty} q(x) \tag{B.3}$$

where

$$q(x) := \begin{cases} \pi & \text{if } -\frac{1}{2} < x < +\frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{B.4}$$

(i.e. $\frac{sin(x)}{x}$ is the Fourier transform of $q(x)$). $\left(\frac{sin(x)}{x}\right)^k$ is the fourier transform of the k-1 – fold self convolution of $q(x)$.

The first values of r_k are:

k	r_k
1	1
2	1
3	3 / 4
4	2 / 3
5	115 / 192
6	11 / 20
7	5887 / 11520
8	151 / 315
9	259723 / 573440
10	15619 / 36288
11	381773117 / 928972800
12	655177 / 1663200
13	20646903199 / 54499737600
14	27085381 / 74131200
15	467168310097 / 1322526965760

Paul Zimmerman found an order 2 recursion for the more general integral

$$c(k,j) = \int_{-\infty}^{\infty} \frac{\sin(x)^k}{x^j} \tag{B.5}$$

It is

$$c(k,j) = k\,\frac{(k-1)\,c(k-2,j-2) - k\,c(k,j-2)}{(j-1)\,(j-2)} \tag{B.6}$$

with

$$c(k,j) = \binom{k-j}{k/2-j/2} \frac{\pi}{2^{k-j}} \quad \text{for} \quad j = 1,2 \tag{B.7}$$

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