

# Generalization of Wrench's identity for Lambert series

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## Abstract

In this short note we give a generalization of an identity for Lambert series by Wrench.

## 1 Wrench's identity

The following relation is given by Knuth (attributed to J. R. Wrench, Jr.) [3, p.644, solution to exercise 5.2.3-27]:

$$W(q) := \sum_{n \geq 1} \frac{a_n q^n}{1 - q^n} = \sum_{n \geq 1} q^{n^2} \left[ a_n + \sum_{k \geq 1} (a_n + a_{n+k}) q^{kn} \right] \quad (1)$$

To verify this identity, write the Lambert sum as

$$W(q) = \sum_{n \geq 1} \sum_{k \geq 1} a_n q^{kn} \quad (2)$$

and write the summands in a table (cf. [4]) as shown in figure 1. Then take sums starting from the diagonal entries  $a_n q^{n^2}$ , taking both the terms to the right in the same row  $\sum_{k \geq 1} a_n q^{n^2 + kn}$ , and the terms below in the same column  $\sum_{j \geq 1} a_{n+j} q^{n^2 + jn}$ . To obtain (1), combine both into a single sum, replacing  $j$  by  $k$  in the column sums.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$\dots$	$k$	$\dots$
$n = 1$	$a_1 q^1$	$a_1 q^2$	$a_1 q^3$	$a_1 q^4$	$a_1 q^5$	$\dots$	$a_1 q^k$	$\dots$
$n = 2$	$a_2 q^2$	$a_2 q^4$	$a_2 q^6$	$a_2 q^8$	$a_2 q^{10}$	$\dots$	$a_2 q^{2k}$	$\dots$
$n = 3$	$a_3 q^3$	$a_3 q^6$	$a_3 q^9$	$a_3 q^{12}$	$a_3 q^{15}$	$\dots$	$a_3 q^{3k}$	$\dots$
$n = 4$	$a_4 q^4$	$a_4 q^8$	$a_4 q^{12}$	$a_4 q^{16}$	$a_4 q^{20}$	$\dots$	$a_4 q^{4k}$	$\dots$
$n = 5$	$a_5 q^5$	$a_5 q^{10}$	$a_5 q^{15}$	$a_5 q^{20}$	$a_5 q^{25}$	$\dots$	$a_5 q^{5k}$	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$
$n$	$a_n q^n$	$a_n q^{2n}$	$a_n q^{3n}$	$a_n q^{4n}$	$a_n q^{5n}$	$\dots$	$a_n q^{nk}$	$\dots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$

Figure 1: The double sum in relation (2) written as a table.

## 2 The generalization

We first derive

$$\sum_{n \geq 1} \frac{a_n q^n}{1 - b_n q^n} = \sum_{n \geq 1} q^{n^2} \left[ a_n b_n^{n-1} + \sum_{k \geq 1} (a_n b_n^{n+k-1} + a_{n+k} b_{n+k}^{n-1}) q^{nk} \right] \quad (3)$$

As before, write as a double sum

$$\sum_{n \geq 1} \sum_{k \geq 1} a_n b_n^{k-1} q^{kn} \quad (4)$$

Take sums starting from the diagonal entries  $a_n b_n^{n-1} q^{n^2}$ . The terms to the right give  $\sum_{k \geq 1} a_n b_n^{n+k-1} q^{n^2+kn}$ , the terms below give  $\sum_{j \geq 1} a_{n+j} b_{n+j}^{n-1} q^{n^2+jn}$ . Write as a single sum to obtain (3).

Replacing  $a_n$  by  $a_n t^n/q^n$  and simplifying gives the desired identity:

$$\begin{aligned} & \sum_{n \geq 1} \frac{a_n t^n}{1 - b_n q^n} = \\ & = \sum_{n \geq 1} q^{n(n-1)} t^n \left[ a_n b_n^{n-1} + \sum_{k \geq 1} (a_n b_n^{n+k-1} q^k + a_{n+k} b_{n+k}^{n-1} t^k) q^{k(n-1)} \right] \quad (5) \end{aligned}$$

For the special case  $a_n = 1/t$  and  $b_n = x/q$  there is Agarwal's relation, see [1] (note the summation starts with  $n = 0$ ),

$$\sum_{n \geq 0} \frac{t^n}{1 - x q^n} = \sum_{n \geq 0} \frac{(1 - x t q^{2n})}{(1 - x q^n)(1 - t q^n)} (x t)^n q^{n^2} \quad (6)$$

and also, see [2],

$$\sum_{n \geq 0} \frac{t^n}{1 - x q^n} = \sum_{n \geq 0} \frac{(q; q)_n}{(x; q)_{n+1} (t; q)_{n+1}} (x t)^n q^{(n^2-n)/2} \quad (7)$$

where  $(z; q)_n = (1-z)(1-zq)(1-zq^2) \cdots (1-zq^{n-1})$  and  $(z; q)_0 = 1$ . Writing identity (5) in a similar way doesn't seem easily possible.

## References

- [1] R. P. Agarwal: Lambert series and Ramanujan, Proceedings of the Indian Academy of Science (Mathematical Sciences), vol.103, no.3, pp.269-293, (December-1993). [2](#)
- [2] Jörg Arndt: On computing the generalized Lambert series, arXiv:1202.6525 [math.CA] (2012). [2](#)
- [3] Donald E. Knuth: The Art of Computer Programming, second edition, Volume 3: Sorting and Searching, Addison-Wesley, (1997). [1](#)
- [4] Thomas J. Osler, Abdul Hassen: On generalizations of Lambert's series, International Journal of Pure and Applied Mathematics, vol.43, no.4, pp.465-484, (2008). [1](#)